

Unit 5

AP Physics 1

Torque and Rotational Dynamics

Rotational kinematics · Linear and rotational motion · Torque · Rotational inertia · Rotational equilibrium · Newton's second law in rotational form

Unit 1
10–15%
Kinematics

Unit 2
18–23%
Force and dynamics

Unit 3
18–23%
Work, energy, power

Unit 4
10–15%
Linear momentum

Unit 5
10–15%
Torque and rotation

Unit 6 5–8%
Rotating systems

Unit 7 5–8%
Oscillations

Unit 8
10–15%
Fluids

Unit 5 is 10–15% of the multiple-choice section. Any of the four free-response questions can use it. Recent ones asked for an experiment that finds a mass by balancing a meterstick, and to compare how fast two toys spin when the same force pulls a string off each.

5.1
Rotational kinematics



5.2
Linear and rotational motion



5.3
Torque



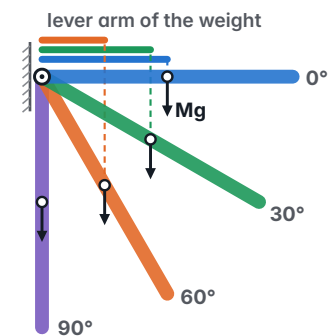
5.4
Rotational inertia



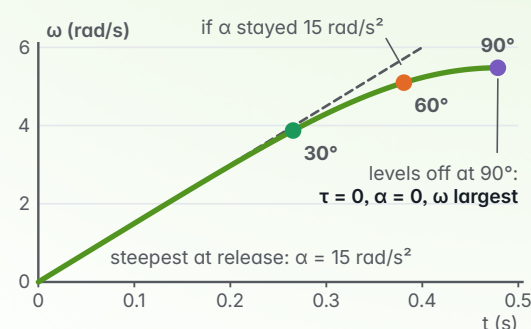
5.5
Rotational equilibrium



5.6
Second law in rotational form



Angle	$r_{\perp}$ (m)	τ (N·m)	α (rad/s ²)	t (s)	ω (rad/s)
0°	0.50	2.0	15	0	0
30°	0.43	1.7	13	0.27	3.9
60°	0.25	1.0	7.5	0.38	5.1
90°	0	0	0	0.48	5.5



A hinged board falling from horizontal. The uniform board is 1.0 m long and 0.40 kg, so $I = \frac{1}{3}ML^2 = 0.133 \text{ kg}\cdot\text{m}^2$, a formula the question gives. Its weight acts at its center, so the weight's lever arm $r_{\perp}$ (the colored bars) is the horizontal distance from the hinge to the center. As the board falls, that distance shrinks, and so do the torque and $\alpha = \tau/I$. The board keeps speeding up, but more and more gently, so its ω - t graph curves away from the dashed line that a constant α would give.

What the exam asks most

- Balance the torques on a beam or rod at rest.** Put your pivot at a support whose force you don't know. A force at the pivot has no lever arm, so it drops out. Then $\Sigma F = 0$ gives that force. (5.5)
- Draw a force diagram that shows where each force acts.** Start each arrow where its force acts on the object: the weight at the center of mass, a string's pull where it is tied, the hinge's push at the hinge. (5.3)
- Find the torque of a force at an angle.** In $\tau = rF \sin \theta$, θ is the angle between the force and the line from the pivot to where the force acts. For a rod 30° below horizontal, the weight makes 60° with the rod: use $\sin 60^\circ$. (5.3)
- Find the acceleration of a block on a pulley.** Write $\Sigma F = ma$ for the block and $\Sigma \tau = I\alpha$ for the pulley, and join them with $a = R\alpha$, where R is the pulley's radius. The tension is less than the block's weight, because the block speeds up as it falls. (5.6)
- Predict how α changes.** $\alpha = \Sigma \tau / I$, so twice the net torque gives twice the α , and twice the I gives half. A small mass twice as far from the axis adds four times as much to I , because it adds mr^2 . (5.4) (5.6)
- Read θ , ω and α graphs.** The slope of the θ - t graph is ω , and the slope of the ω - t graph is α . The constant- α equations need a constant net torque. A falling rod's torque shrinks, so its ω - t graph curves. (5.1)

Not on the exam

The right-hand rule for which way a torque points, and objects turning about two axes at once. You don't need to remember I for a rod, hoop or disk: the question gives it.

Formulas for this unit

The equation sheet you get on exam day lists these:

- $\omega = \omega_0 + \alpha t$, $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ and $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$, for constant α only.
- $s = r\theta$, $v = r\omega$ and $a_T = r\alpha$, with θ in radians.
- $\tau = r_{\perp}F = rF \sin \theta$, and $\alpha_{\text{sys}} = \Sigma \tau / I_{\text{sys}}$ for the whole turning system.
- $I = \Sigma m_i r_i^2$ and $I' = I_{\text{cm}} + Md^2$, with d measured from the center of mass.

It doesn't list these, so learn them:

- $\omega_{\text{avg}} = \Delta \theta / \Delta t$ and $\alpha_{\text{avg}} = \Delta \omega / \Delta t$.
- An object that stays at rest has $\Sigma F = 0$ and $\Sigma \tau = 0$.

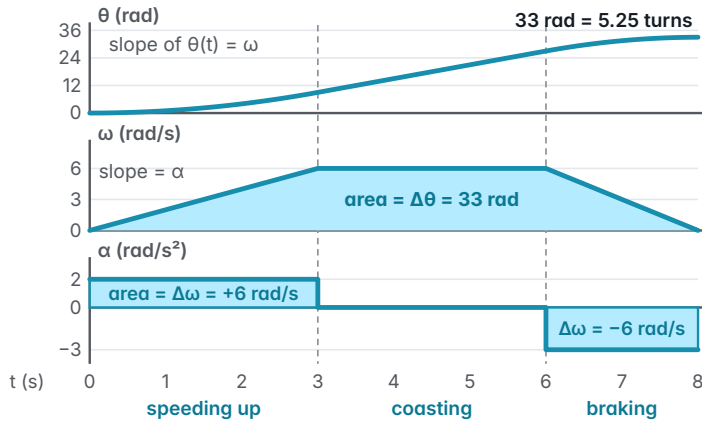
Exam answers use $g = 10 \text{ m/s}^2$, as this sheet does; correct work with 9.8 (on the equation sheet) or 9.81 also gets full credit.

Rotation works like motion in a line

Each quantity has a partner from Units 1 and 2: θ for x , ω for v , α for a , torque for force and I for mass. So $\Sigma \tau = I\alpha$ plays the part of $\Sigma F = ma$.

5.1 Rotational kinematics

Call one way of turning positive, usually counterclockwise, and measure angles in **radians**. Every point of a rigid object turns through the same angle, so one $\Delta\theta$, ω and α describe the whole object. When ω and α have the same sign it speeds up; when the signs differ it slows down.



Equations for turning

On the equation sheet, for a constant α (a constant net torque) only:

$$\omega = \omega_0 + \alpha t \quad \text{leaves out } \Delta\theta$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \quad \text{leaves out final } \omega$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \quad \text{leaves out } t$$

Use the one that leaves out the quantity you neither know nor need. Know the averages by heart: $\omega_{\text{avg}} = \Delta\theta/\Delta t$, $\alpha_{\text{avg}} = \Delta\omega/\Delta t$.

Change to radians first

1 rev = 2π rad = 360° . Multiply revolutions per minute (rpm) by $2\pi/60$: a fan at 600 rpm turns at 62.8 rad/s. One turn every T seconds is $\omega = 2\pi/T$. Only an angle inside sin or cos stays in degrees.

Worked example

A fan turning at 30 rad/s brakes evenly and stops after turning through 50 rad. Find α and the stopping time.

- No time is given, so use $\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$: $0 = 30^2 + 2\alpha(50)$.
- $\alpha = -900/100 = -9.0$ rad/s², negative because it slows.
- $t = \Delta\omega/\alpha = (0 - 30)/(-9.0) = 3.3$ s.

$$\alpha = -9.0 \text{ rad/s}^2, t = 3.3 \text{ s}$$

Exam trap

- Using the constant- α equations when the torque changes. They need a fixed α . The board on page 1 starts at 15 rad/s², but its torque shrinks as it falls, so assuming 15 rad/s² all the way gives $\omega = \sqrt{2(15)(\pi/2)} = 6.9$ rad/s at the bottom, not the real 5.5. Instead, sketch ω with a shrinking slope, like the graph on page 1.

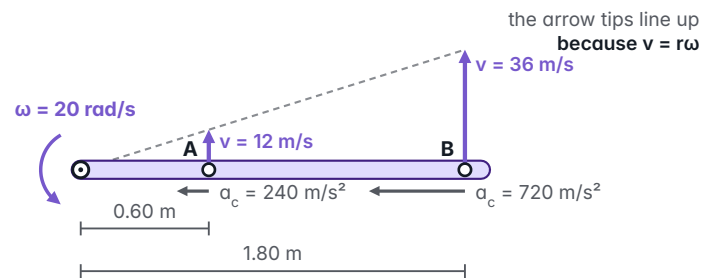
Quick check

A wheel starts from rest with $\alpha = 3.0$ rad/s². How many turns does it make in 2.0 s?

Answer: $\Delta\theta = \frac{1}{2}(3.0)(2.0)^2 = 6.0$ rad, and $6.0/2\pi = 0.95$ rev.

5.2 Linear and rotational motion

Every point of a rigid object shares the same $\Delta\theta$, ω and α . But a point twice as far from the axis travels twice as far **along its circle**, at twice the speed and with twice the tangential acceleration.



These arrows are velocities and accelerations, not forces.

Equations for a point on a turning object

$$s = r\theta \quad \text{arc length}$$

$$v = r\omega \quad \text{speed along the circle}$$

$$a_T = r\alpha \quad \text{acceleration along the path}$$

$$a_c = \frac{v^2}{r} = r\omega^2 \quad \text{acceleration toward the axis}$$

All of these need θ in radians. All are on the equation sheet except $r\omega^2$, which you get by putting $v = r\omega$ into v^2/r .

Tangential acceleration a_T

It points along the path and changes the speed. It is zero when $\alpha = 0$.

Centripetal acceleration a_c

It points toward the axis and changes the direction of motion whenever the object turns.

Worked example

A fan blade turns at 20 rad/s. A is 0.60 m from the axis, B is 1.80 m. Compare their speeds and centripetal accelerations.

- Same ω for both. $v_A = (0.60)(20) = 12$ m/s and $v_B = (1.80)(20) = 36$ m/s.
- $a_c = r\omega^2$: $(0.60)(20)^2 = 240$ m/s² and $(1.80)(20)^2 = 720$ m/s².

B has three times the speed and three times the a_c

Exam trap

- Saying the outer point turns faster. Every point turns through the same angle in the same time, so all share one ω . Say instead that B has the larger *speed*: 36 m/s against A's 12 m/s.
- Putting degrees into $s = r\theta$. A point 1.80 m from the axis that goes a quarter turn travels $1.80 \times \pi/2 = 2.83$ m. Using 90 for θ gives 162 m. Convert first: $90^\circ \times \pi/180 = \pi/2$ rad.

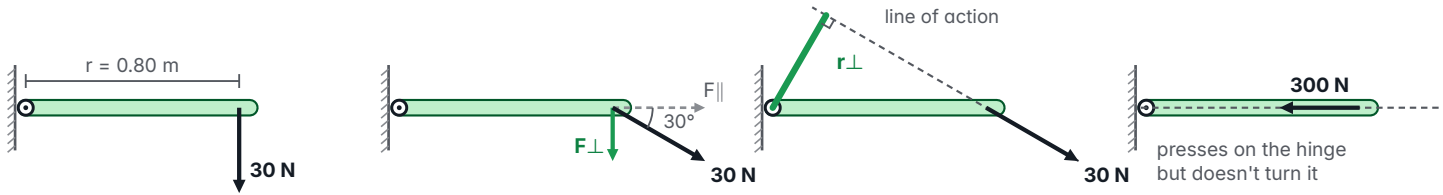
Quick check

Two children ride a merry-go-round, one 1.0 m and one 2.0 m from the center. Compare their ω , v and a_c .

Answer: same ω ; the outer child has **twice the v** and **twice the a_c** ($a_c = r\omega^2$).

5.3 Torque

Torque is force times lever arm



1 Push at 90° to the door
 $\tau = rF = (0.80)(30) = 24 \text{ N}\cdot\text{m}$

2 Push at 30°, using $F_{\perp}$
 $F_{\perp} = F \sin \theta = 15 \text{ N}$, $\tau = 12 \text{ N}\cdot\text{m}$

3 Same push, using the lever arm
 $r_{\perp} = r \sin \theta = 0.40 \text{ m}$, $\tau = 12 \text{ N}\cdot\text{m}$

4 Aimed at the hinge
 $\theta = 180^\circ$, $r_{\perp} = 0$, so $\tau = 0$

A door seen from above, with the hinge on the left and the handle 0.80 m out. In panel 2 only $F_{\perp}$, the part of the push perpendicular to the door, turns it; $F_{\parallel}$ pulls along the door and makes no torque. Panel 3 gets the same 12 N·m from the whole push times its lever arm, the perpendicular distance from the hinge to the force's line of action.

The definition of torque

$$\tau = r_{\perp} F = rF \sin \theta \text{ (on the equation sheet)}$$

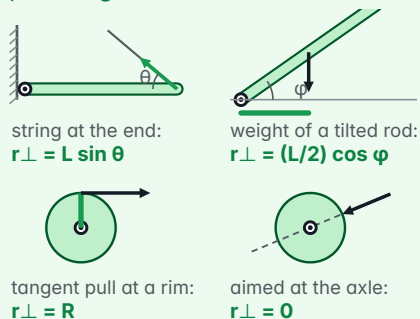
- r is the distance from the axis to the point where the force acts.
- θ is the angle between $\vec{r}$ and $\vec{F}$, drawn tail to tail.
- $r_{\perp}$, the **lever arm**, is the perpendicular distance from the axis to the force's **line of action**.
- Torque is in N·m. Don't write joules, because torque is not energy.

Know by heart

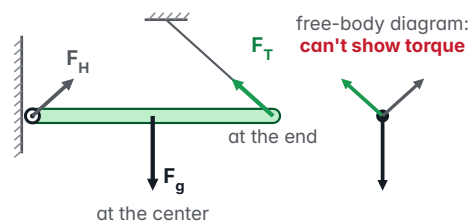
- Say which axis you take torques about. The same force has a different torque about each point.
- Call counterclockwise positive. A torque's sign is the way that force alone would turn the object about the axis.
- A force whose line of action passes through the axis has **zero torque**, however big it is.

If you change	Torque
Force $\times 2$	$\times 2$
Distance $\times 2$, same angle	$\times 2$
Angle $90^\circ \rightarrow 30^\circ$	$\times \frac{1}{2}$
Force aimed at the axis	0

Finding the lever arm



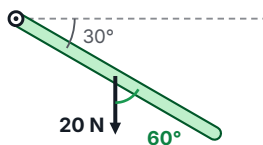
Force diagrams for objects that turn



A rod on a wall hinge, held up by a slanted string. F_g is the weight, F_T the string's tension and F_H the hinge's force, each drawn where it acts.

Worked example

A uniform 2.0 kg, 1.2 m rod pivots at its left end, held 30° below horizontal. Find the torque of its weight about the pivot.



- $r = 0.60 \text{ m}$ and $F = Mg = 20 \text{ N}$.
- The angle between $\vec{r}$ (along the rod) and $\vec{F}$ (down) is 60° , not 30° .
- $\tau = (0.60)(20) \sin 60^\circ = 10.4 \text{ N}\cdot\text{m}$.

$\tau = 10 \text{ N}\cdot\text{m}$ to two significant figures; $\sin 30^\circ$ would give $6.0 \text{ N}\cdot\text{m}$

Explaining which torque is larger

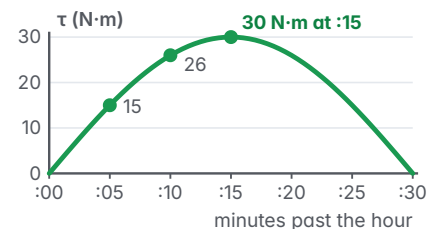
The pushes in panels 1 and 3 are both 30 N, but the **line of action** in panel 1 is farther from the hinge, so it has the longer **lever arm** and the larger **torque**: 24 N·m against 12 N·m.

Quick check

A 40 N force acts 0.50 m from an axis at 37° to $\vec{r}$. Find τ , then double the force and halve the distance.

Answer: $(0.50)(40)(0.60) = 12 \text{ N}\cdot\text{m}$ (the equation sheet gives $\sin 37^\circ = 0.60$). Doubling F and halving r leaves **12 N·m**.

How torque changes with the angle



A 10 N bird sits 3.0 m out on a clock's minute hand: $\tau = (3.0)(10) \sin \phi$, where ϕ is the hand's angle from straight up.

Worked example

On a wheel with a fixed axle, one rope pulls 20 N counterclockwise along the rim ($R = 0.30 \text{ m}$) and another pulls 50 N clockwise along the small inner hub ($r = 0.10 \text{ m}$). Find the net torque.

- Counterclockwise is positive. Each rope's lever arm is its radius.
- $\Sigma \tau = +(0.30)(20) - (0.10)(50) = 6.0 - 5.0$

$\Sigma \tau = +1.0 \text{ N}\cdot\text{m}$, counterclockwise

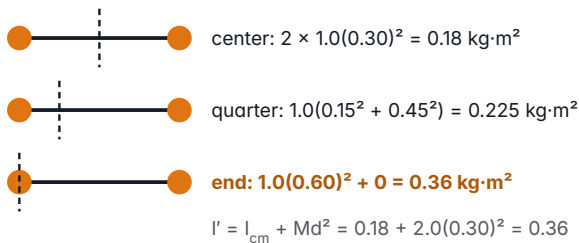
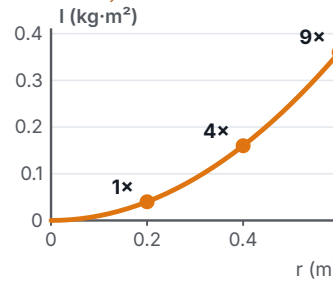
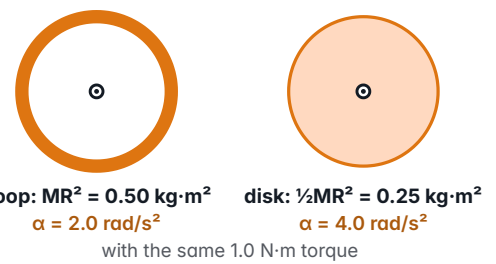
Exam trap

- Using the distance to the force as the lever arm. They match only when the force is perpendicular to $\vec{r}$. In panel 3 the push is 0.80 m from the hinge, but its lever arm is $0.80 \sin 30^\circ = 0.40 \text{ m}$: $\tau = 12 \text{ N}\cdot\text{m}$, not 24.
- Drawing a force at the wrong point. Where a force acts sets its lever arm. Drawing the tilted rod's weight at its tip instead of its center doubles its torque to 20.8 N·m; draw it at the center.
- Leaving the calculator in radian mode. The angle inside $\sin$ is usually in degrees, and in radian mode $\sin 30$ gives -0.988 instead of 0.500. Page 7 shows how to switch. Angles in $s = r\theta$ and the ω and α equations still go in radians.

5.4 Rotational inertia

The farther the mass is from the axis, the harder the object is to spin up

Two 1.0 kg masses 0.60 m apart, spun about three axes

One bead at r , $2r$ and $3r$: $I = mr^2$ Same $M = 2.0 \text{ kg}$, $R = 0.50 \text{ m}$ 

Rotational inertia I measures how hard an object is to spin up, the way mass measures how hard it is to speed up in a line. It depends on the mass and on each piece's distance from the axis, squared. A question gives you the hoop and disk formulas when it needs them.

Equations for rotational inertia

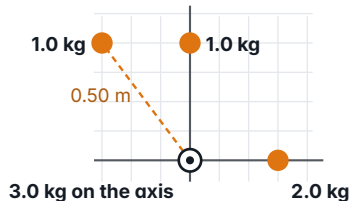
$$I = \sum m_i r_i^2 \quad \text{point masses}$$

$$I' = I_{\text{cm}} + Md^2 \quad \text{moving the axis}$$

Both are on the equation sheet. I is in $\text{kg}\cdot\text{m}^2$, and r is each mass's perpendicular distance from the axis. I changes only when mass is added or moved or the axis changes, never with how fast the object spins.

Worked example

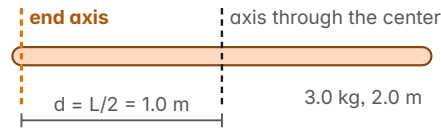
Four small masses lie in a plane; the axis runs through the origin, perpendicular to the page. Grid spacing is 0.10 m. Find I .



- Squared distances: 0.09, 0.16, 0.25 and 0 m^2 .
- $I = 2.0(0.09) + 1.0(0.16) + 1.0(0.25) + 3.0(0)$

$I = 0.59 \text{ kg}\cdot\text{m}^2$ The 3.0 kg mass on the axis adds nothing.

Spinning about a different axis



Worked example

A uniform 3.0 kg, 2.0 m rod has $I_{\text{cm}} = \frac{1}{12}ML^2$ about its center (given). Find I about one end.

- $I_{\text{cm}} = \frac{1}{12}(3.0)(2.0)^2 = 1.0 \text{ kg}\cdot\text{m}^2$
- $I' = 1.0 + (3.0)(1.0)^2 = 4.0 \text{ kg}\cdot\text{m}^2$, the same as $\frac{1}{3}ML^2$.

$I_{\text{end}} = 4.0 \text{ kg}\cdot\text{m}^2$, four times I_{cm}

Using $I' = I_{\text{cm}} + Md^2$

- I_{cm} is I about the axis through the center of mass (cm), and I' is I about a parallel axis a distance d from it.
- M is the whole object's mass.
- Md^2 is never negative, so I about any other parallel axis is bigger than I_{cm} .

Rotational inertias the question gives you

Hoop or thin ring, center	MR^2
Solid disk or cylinder, center	$\frac{1}{2}MR^2$
Uniform rod, center	$\frac{1}{12}ML^2$
Uniform rod, one end	$\frac{1}{3}ML^2$

You don't need to memorize these, but it helps to see why they rank this way: the more of the mass sits far from the axis, the bigger the number in front.

Quick check

A 2.0 kg disk of radius 0.10 m ($I_{\text{cm}} = \frac{1}{2}MR^2$) spins about an axle through its rim, parallel to its central axis. Find I .

Answer: $0.010 + (2.0)(0.10)^2 = 0.030 \text{ kg}\cdot\text{m}^2$, three times I_{cm} .

Why a hoop is slower to spin up

All of the hoop's mass is a distance R from the axis, while much of the disk's mass is closer to the axis. So the hoop has the larger rotational inertia, and the same torque gives it the smaller angular acceleration.

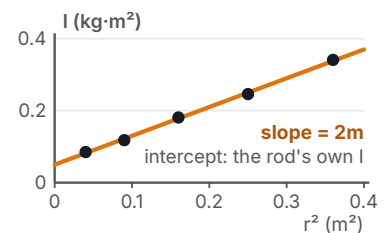
"Same mass, so same I " earns no credit, because I depends on where the mass is.

The tightrope walker's pole

A long pole puts mass far from the wire, which is the axis a walker would tip about. That raises I , so a gust's torque gives a smaller α and the walker has more time to recover. The same mass held close to the body would barely help.

Experiment: I against r^2

Slide two equal masses m out along a rotating rod, measure I at each position (page 6 shows how), and plot I against r^2 .



Quick check

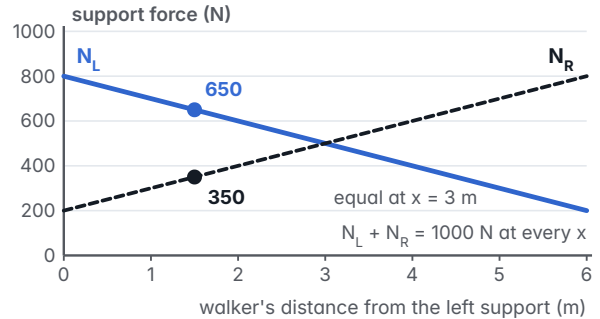
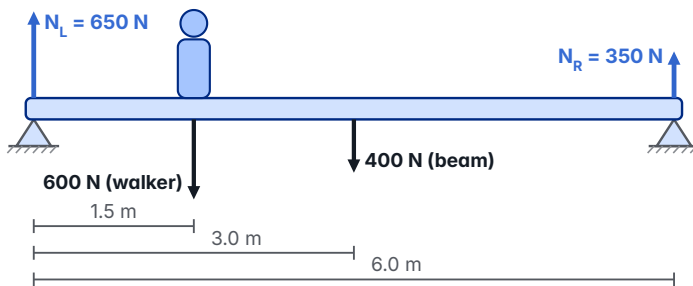
A 0.50 kg bead slides from 0.20 m to 0.60 m along a light spinning rod. By what factor does its I change? Hoop or disk of equal M and R : which spins up faster under the same torque?

Answer: $(0.60/0.20)^2 = 9$ times. The disk: smaller I , bigger α .

Exam trap

- Forgetting to square r . Moving a bead from 0.20 m to 0.40 m doesn't double its I . Since $(0.40/0.20)^2 = 4$, it quadruples.
- Measuring r from the wrong point. r is measured from the axis, not from the object's center: a bead at the tip of a 1.0 m rod spun about its other end has $r = 1.0$ m. Using 0.50 m makes its I four times too small.
- Using the diameter as R . A wheel 1.0 m across has $R = 0.50$ m; using 1.0 m makes MR^2 four times too big.

5.5 Rotational equilibrium

With zero net torque, ω stays constant

A 40 kg, 6.0 m walkway on two supports, with a 60 kg walker 1.5 m from the left end. N_L and N_R are the supports' upward forces. Taking torques about the left support leaves N_L out, because its lever arm is zero: $N_R(6.0) = 400(3.0) + 600(1.5)$, so $N_R = 350$ N. Then $\Sigma F = 0$ gives $N_L = 1000 - 350 = 650$ N.

Conditions for staying at rest

$$\Sigma \vec{F} = 0 \text{ and } \Sigma \tau = 0$$

Neither is on the equation sheet. (The page 1 board is at rest at release, but its net torque starts it turning.) Zero net torque means $\alpha = 0$ in $\alpha_{\text{sys}} = \Sigma \tau / I_{\text{sys}}$, so ω stays constant. At rest, the torques cancel about every axis, so pick the most useful one.

Why the lines are straight

With the beam's weight $W_b = 400$ N and the walker's $W_w = 600$ N at x , torques about the left support give $N_R L = W_b \frac{L}{2} + W_w x$, so

$$N_R = \frac{W_b}{2} + \frac{W_w}{L} x$$

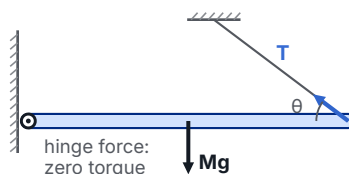
That is a straight line with intercept $W_b/2 = 200$ N and slope $W_w/L = 100$ N/m. A walker twice as heavy doubles the slope but not the intercept.

Solving a balance problem

- 1 Draw a force diagram with each force where it acts, and mark the distances.
- 2 Take torques about a point where an **unknown force** acts, often the hinge, so that force drops out.
- 3 Call counterclockwise positive, write each torque as $\pm$ force $\times$ lever arm, and set $\Sigma \tau = 0$.
- 4 Then $\Sigma F_x = 0$ and $\Sigma F_y = 0$ give the support forces.
- 5 Check a special case: a walker at $x = 0$ stands over the left support, so N_R should be half the beam's weight.

Worked example

A uniform 2.0 kg beam is hinged to a wall and held level by a string from its far end to a hook above, at angle θ to the beam. Find the tension.



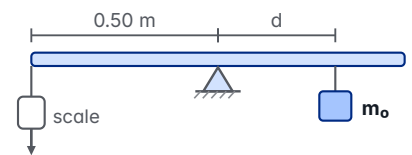
- 1 About the hinge: $T L \sin \theta = Mg \frac{L}{2}$, so $T = \frac{Mg}{2 \sin \theta}$. L cancels.
- 2 With $Mg = 20$ N, T is 20 N at 30° , 16.7 N at 37° , 12.5 N at 53° and 10 N at 90° .

A shallower string needs more tension

Explaining the trend in words

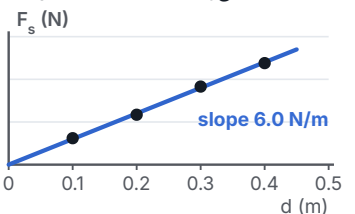
At a **smaller** θ , the string's lever arm $L \sin \theta$ is shorter, so to balance the same torque from the beam's weight, the **tension must be larger**.

Experiment: an unknown mass

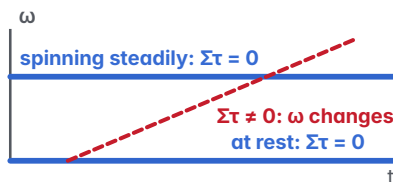


Pivot the stick at its center so its weight has no torque. Hang the block (unknown mass m_0) at distance d and read the force F_s on the spring scale holding the other end level, 0.50 m from the pivot. Repeat for several d . Torques about the pivot give

$$F_s(0.50 \text{ m}) = m_0 g d$$



Plot F_s against d : a line through the origin with slope $m_0 g / (0.50 \text{ m})$. A slope of 6.0 N/m gives $m_0 = (0.50)(6.0)/10 = 0.30$ kg.

 ω with and without a net torque $\Sigma F = 0$ but $\Sigma \tau \neq 0$

Hands push opposite ways on a steering wheel. The forces cancel, but the torques add, so it starts to turn.

 $\Sigma \tau = 0$ but $\Sigma F \neq 0$

A plank dropped level doesn't start to spin, but it speeds up as it falls.

Quick check

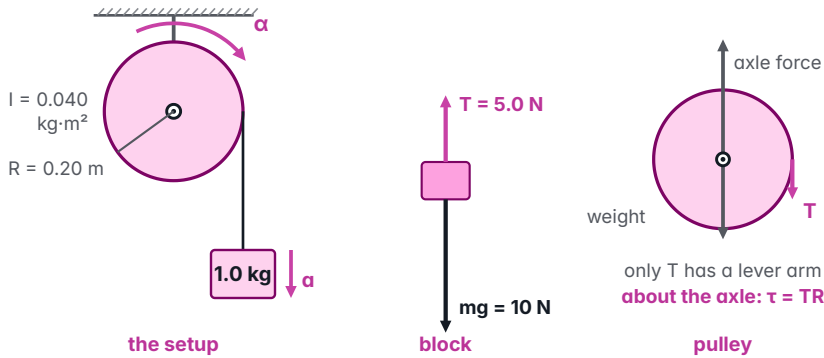
A 30 kg child sits 1.5 m from the pivot of a uniform seesaw pivoted at its center. Where must a 45 kg child sit to balance? Does tilting the board upset the balance?

Answer: $30(1.5) = 45d$, $d = 1.0$ m. No: tilting shrinks both lever arms by the same factor.

Exam trap

- **Balancing forces where you need to balance torques.** The scale and the block sit at different distances from the pivot, so $F_s = m_0 g$ (3.0 N at every d) is wrong. Torques give $F_s(0.50) = m_0 g d$: 1.5 N at $d = 0.25$ m.
- **Saying the scale measures torque.** A spring scale reads a force in newtons. Multiply by the lever arm to get torque: 3.0 N at 0.50 m is 1.5 N·m.
- **Thinking a turning object needs a net torque to keep turning.** A fan spinning at a steady 20 rad/s has zero net torque. For a steady spin, set $\Sigma \tau = 0$: the motor's torque just balances friction's.

5.6 Newton's second law in rotational form

The net torque divided by I gives α 

⚡ Three equations, solved together

Block: $mg - T = ma$ from $\vec{a}_{\text{sys}} = \Sigma \vec{F} / m_{\text{sys}}$

Pulley: $TR = I\alpha$ from $\alpha_{\text{sys}} = \Sigma \tau / I_{\text{sys}}$

The string doesn't slip: $a = R\alpha$ from $a_T = r\alpha$

All three start from the equation sheet. The pulley gives $T = I\alpha/R = Ia/R^2$; put that into the block's equation, $mg - (I/R^2)a = ma$, so

$$a = \frac{mg}{m + I/R^2} = \frac{10}{1.0 + 1.0} = 5.0 \text{ m/s}^2$$

The pulley's I/R^2 acts like extra mass the block has to drag along. Then $T = m(g - a) = 5.0 \text{ N}$ and $\alpha = a/R = 25 \text{ rad/s}^2$. Check: TR and $I\alpha$ both equal $1.0 \text{ N}\cdot\text{m}$.

A 1.0 kg block unwinds a light string from a fixed pulley ($R = 0.20 \text{ m}$, $I = 0.040 \text{ kg}\cdot\text{m}^2$, frictionless axle, no slip). The tension, 5.0 N, is less than the block's 10 N weight because the block speeds up.

📖 Newton's second law for rotation

$$\alpha_{\text{sys}} = \frac{\Sigma \tau}{I_{\text{sys}}} = \frac{\tau_{\text{net}}}{I_{\text{sys}}}$$

on the equation sheet

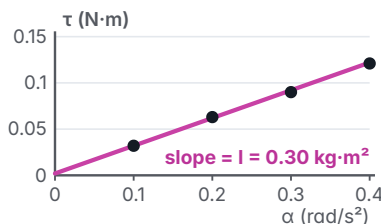
- Twice the net torque gives twice the α ; twice the I gives half the α .
- α has the same sign as the **net torque**. If ω has the opposite sign, the object slows down.
- Take the torques and I about the **same axis**.
- "sys" means the whole system that turns together.

📖 Where $\alpha = \Sigma \tau / I$ comes from

Push a small mass m at radius r along its circle: $F = ma_T = m(r\alpha)$. Multiply both sides by r to get $rF = (mr^2)\alpha$, which is $\tau = I\alpha$.

🔍 Experiment: I from a graph

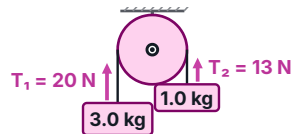
Pull a string wrapped around the axle (radius r) with a spring scale held at a steady reading F , so $\tau = Fr$. Time the wheel from rest through a measured angle (count turns of a mark on the rim) and get α from $\Delta\theta = \frac{1}{2}\alpha t^2$. Repeat for several F and plot τ against α ; the slope is I .



Friction in the axle shifts the whole line up, and the intercept on the τ axis is the friction torque.

📖 Worked example

Blocks of $m_1 = 3.0 \text{ kg}$ and $m_2 = 1.0 \text{ kg}$ hang from a string over a pulley with $I = 0.020 \text{ kg}\cdot\text{m}^2$, $R = 0.10 \text{ m}$. Find a and both tensions.



- 1 The pulley adds $I/R^2 = 2.0 \text{ kg}$ to the mass being moved: $a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2} = \frac{20}{6.0} = 3.3 \text{ m/s}^2$.
- 2 $T_1 = m_1(g - a) = 20 \text{ N}$; $T_2 = m_2(g + a) = 13 \text{ N}$.
- 3 Check: $(T_1 - T_2)R$ and $I(a/R)$ both equal $0.67 \text{ N}\cdot\text{m}$.

$$a = 3.3 \text{ m/s}^2, T_1 = 20 \text{ N}, T_2 = 13 \text{ N}$$

📖 Worked example

A spool (a solid disk, $I = \frac{1}{2}mR^2$) falls as its string unwinds from its outer rim, held by a fixed hand. Find a and T .

- 1 $mg - T = ma$; $TR = \frac{1}{2}mR^2\alpha$; $a = R\alpha$.
- 2 So $T = \frac{1}{2}ma$ and $mg = \frac{3}{2}ma$.

$$a = \frac{2}{3}g = 6.7 \text{ m/s}^2, T = \frac{1}{3}mg$$

Three shapes of the ω - t graph

📖 Worked example

The hinged board from page 1 (1.0 m, 0.40 kg, $I = \frac{1}{3}ML^2$) is released from horizontal. Then a 0.40 kg lump is taped to its tip. Compare the starting α .

- 1 Board: $\tau = (4.0)(0.50) = 2.0 \text{ N}\cdot\text{m}$, $I = 0.133 \text{ kg}\cdot\text{m}^2$, $\alpha = 15 \text{ rad/s}^2$.
- 2 With the lump: $\tau = 2.0 + (4.0)(1.0) = 6.0 \text{ N}\cdot\text{m}$, three times the board's, and $I = 0.133 + 0.40 = 0.533 \text{ kg}\cdot\text{m}^2$, four times.

$$\alpha = 6.0/0.533 = 11 \text{ rad/s}^2: \text{ less than the bare board's } 15$$

⚠ Exam trap

- **Setting the tension equal to mg .** The block speeds up downward, so the net force on it points down and T is less than mg : above, $T = 5.0 \text{ N}$, not 10 N. Write $mg - T = ma$ instead.
- **Using one tension on both sides of a pulley that has mass.** A pulley that speeds up needs a net torque, so label the tensions T_1 and T_2 and use $(T_1 - T_2)R = I\alpha$: 20 N and 13 N in the two-block example.
- **Putting the largest α and the largest ω at the same place.** The board on page 1 has its largest α , 15 rad/s^2 , at release, where the lever arm is longest, and its largest ω , 5.5 rad/s , at the bottom, where α is zero.

🔍 Quick check

The pulley at the top of this page has $I/R^2 = 1.0 \text{ kg}$. If the block's mass doubles to 2.0 kg, does a double?

Answer: no. $a = 20/(2.0 + 1.0) = 6.7 \text{ m/s}^2$, up from 5.0: I/R^2 stays in the denominator.

Torque on the free-response questions

Start from a law on the equation sheet and show every step

When you are asked to	What to write	How students lose the point
Derive an expression (5.6)	Start from a law on the equation sheet, such as $\alpha_{\text{sys}} = \Sigma \tau / I_{\text{sys}}$ or $\vec{a} = \Sigma \vec{F} / m$. Write each torque as force $\times$ lever arm, add any link such as $\alpha = R\alpha$ or $\Delta\theta = \ell/r$ (a string of length ℓ unwinding from radius r), and solve using only the symbols the question lists.	They replace the tension with mg , or start from a memorized shortcut instead of a law.
Draw a force diagram of a rod, beam or pulley (5.3)	Draw one arrow per force, each starting where that force acts: the weight at the center of mass, a string's pull where it is tied, the hinge's force at the hinge. Label each one (F_g , F_T , F_H) and make the lengths roughly right.	All the arrows start at one dot, the weight is labeled just "g", or the arrows couldn't balance even though the object is at rest.
Describe a procedure to find a mass or I (5.5) (5.4)	Name each instrument and what it measures: a spring scale for force, a meterstick for distance, a stopwatch for time. Say which quantity you change, over several values, repeat each trial, and say which graph's slope gives the answer.	The plan says the scale measures "torque", or never says what changes from one trial to the next.
Graph the data and use a best-fit line (5.5)	Choose axes that make the relation a straight line, such as τ against α . Label each axis with its quantity and unit, draw one straight line through the trend, take the slope from two points on that line, then turn the slope into the answer.	They join the dots instead of drawing one line, or report the slope without turning it into the mass or I .
Sketch ω against t for a released rod, or say where α is greatest (5.1)	From rest, ω rises with a slope that shrinks to zero at the bottom. α is greatest at release from horizontal, where the weight's lever arm is longest, and ω is greatest at the bottom.	They draw a straight line, as if α were constant, or put the largest α and the largest ω at the same place.

⊙ A short free-response question, worked

Two tops sit at rest on fixed vertical axes with no friction, and both have the same rotational inertia I . A string of length ℓ is wrapped around each top's stem, and the same constant force F pulls each string off. Top 2's stem is twice as wide as top 1's. Part A asks which top spins faster when its string comes off, with reasoning in words; part B asks you to derive ω ; part C asks whether the two agree.

Part	An answer that earns the points	What earns them, and what loses them
A	They spin equally fast. Top 2's wider stem gives its string twice the lever arm, so top 2 gets twice the torque and twice the angular acceleration. But the same length of string wraps fewer times around a wider stem, so top 2 turns through only half the angle. Twice the angular acceleration over half the angle ends at the same angular speed.	Say what is the same for both, then what differs, then the result. "Top 2 has more torque, so it spins faster" ignores that top 2 turns through half the angle, and loses the point.
B	$\alpha = \frac{\Sigma \tau}{I} = \frac{Fr}{I}$ and $\Delta\theta = \frac{\ell}{r}$, so $\omega^2 = 0 + 2\alpha\Delta\theta = 2 \cdot \frac{Fr}{I} \cdot \frac{\ell}{r}$ and $\omega = \sqrt{\frac{2F\ell}{I}}$.	Begin with a law from the equation sheet, show each substitution, and give the answer in the symbols the question lists (F , ℓ , I).
C	The radius r cancels, so the width of the stem doesn't change ω . That agrees with part A.	Point to the feature of your expression (here, no r) and say how it matches part A. Repeating the answer from part A earns nothing.

With numbers: $F = 4.0 \text{ N}$, $\ell = 0.60 \text{ m}$, $I = 1.2 \times 10^{-4} \text{ kg}\cdot\text{m}^2$ and stems of radius 1.0 cm and 2.0 cm . Top 1 gets $\alpha = 333 \text{ rad/s}^2$ for 60 rad and top 2 gets 667 rad/s^2 for 30 rad , and both end at 200 rad/s . The same expression shows that a top with half the I would end $\sqrt{2}$ times faster, at 283 rad/s .

☰ Set degree mode

TI-84 Plus CE. **mode**, arrow down to the row RADIAN DEGREE, highlight DEGREE, press **enter**, then **2nd mode** to quit.

TI-Nspire CX. **on** for the home screen, then 5 Settings, 2 Document Settings. Set Angle to Degree, move to Make Default and press **enter**.

Desmos. Open the wrench (graph settings) and choose Degrees.

⊙ Plots that give a straight line

Relation	Plot	Slope
$\Sigma \tau = I\alpha$	τ against α	I
$F_s r_s = m_0 g d$	F_s against d	$m_0 g / r_s$
$T = \frac{Mg}{2 \sin \theta}$	T against $\frac{1}{\sin \theta}$	$Mg/2$
$I = I_0 + 2mr^2$	I against r^2	$2m$
$\omega^2 = 2\alpha\Delta\theta$	ω^2 against $\Delta\theta$	2α
$v = r\omega$	v against r	ω

Choose the axes so that the slope is the quantity you want. If the quantity you vary sits in a denominator, plot against 1 over it, as with $1/\sin \theta$. Here r_s is the scale's distance from the pivot and I_0 is the rod's own I .

🔗 Unit 6 builds on this unit

Unit 6 uses τ , I and ω for energy and angular momentum. These are all on the equation sheet:

$$K = \frac{1}{2} I \omega^2 \text{ and } W = \tau \Delta\theta$$

$$L = I\omega \text{ and } \Delta L = \tau \Delta t$$

$$\text{Rolling without slipping: } \Delta x_{\text{cm}} = r \Delta\theta$$

🔍 Checking your answer

- Torque is in N·m (never J), I in $\text{kg}\cdot\text{m}^2$ and α in rad/s^2 .
- α has the same sign as the net torque.
- For a block falling from a pulley that has mass, the tension must come out between 0 and mg .
- An object that stays at rest needs both $\Sigma \vec{F} = 0$ and $\Sigma \tau = 0$. If you used only one, you haven't finished.