

# Unit 6

AP Physics 1

## Energy and Momentum of Rotating Systems

Rotational kinetic energy · Torque and work · Angular momentum and angular impulse · Conservation of angular momentum · Rolling · Orbiting satellites

**Unit 1**  
10–15%  
Kinematics

**Unit 2**  
18–23%  
Force and dynamics

**Unit 3**  
18–23%  
Work, energy, power

**Unit 4**  
10–15%  
Linear momentum

**Unit 5**  
10–15%  
Torque and rotation

**Unit 6** 5–8%  
Rotating systems

**Unit 7** 5–8%  
Oscillations

**Unit 8**  
10–15%  
Fluids

**Unit 6 is 5–8% of the multiple-choice section.** Its ideas turn up in free-response questions much more often than 5–8% suggests: recent questions asked about a string pulled off a spinning toy, two objects given the same torque for the same time, and a rod swinging on a pivot.

**6.1**  
Rotational kinetic energy

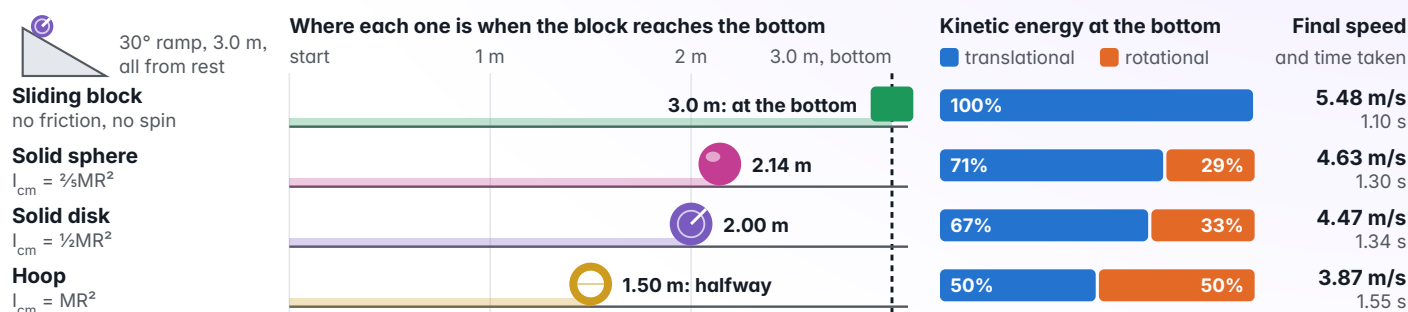
**6.2**  
Torque and work

**6.3**  
Angular momentum and angular impulse

**6.4**  
Conservation of angular momentum

**6.5**  
Rolling

**6.6**  
Motion of orbiting satellites



Four objects start from rest at the top of a 30° ramp 3.0 m long, a 1.5 m drop ( $g = 10 \text{ m/s}^2$ ); the sphere, disk and hoop roll without slipping. Gravity does the same work per kilogram on each, but a rolling object has to put part of that energy into spinning, so less is left for moving down the ramp and it falls behind. Mass and radius cancel out. The order depends only on  $I_{cm}/MR^2$ , which is bigger when more of the mass sits far from the center.

### What the exam asks most

- Which of two spinning objects ends up faster.** First ask what the two have in common. If the same force pulls through the same distance, it does the same work, so both gain the same kinetic energy and the one with less rotational inertia spins faster. If the same torque acts for the same time, both gain the same angular momentum. (6.2) (6.3)
- A ball, disk or hoop rolling down a ramp.** Use  $Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$  with  $\omega = v/R$ . It ends slower than a frictionless block, because part of its energy goes into spinning. (6.5)
- A spin that changes with no outside torque.** A skater pulls in their arms, or clay sticks to a spinning rod. Set  $I_i\omega_i = I_f\omega_f$  and say why no outside torque acts. Kinetic energy changes: it rises when the skater pulls in and falls when the clay sticks. (6.4)
- A rod swinging down on a pivot.** Gravity's work is  $Mg$  times how far the center of mass drops: for a uniform rod swinging down from horizontal, that is  $L/2$ , not  $L$ . Gravity's torque, and so the angular acceleration, is largest when the rod is horizontal. (6.2)
- A satellite in orbit.** Its total energy and its angular momentum stay constant, so on an ellipse it moves fastest when it is closest to the planet. Be ready to derive  $v = \sqrt{GM/r}$  for a circular orbit from  $GMm/r^2 = mv^2/r$ . (6.6)
- Torque and angular momentum graphs.** The area under a torque-angle graph is the work. The area under a torque-time graph is the change in angular momentum, and the slope of an angular momentum-time graph is the net torque. (6.3)

### Not on the exam

The right-hand rule for the direction of angular momentum, calculations for a ball that skids before it rolls, and Kepler's first and second laws by name.

### The equations for this unit

These are on the exam's equation sheet, each next to its straight-line version:

$$K = \frac{1}{2}I\omega^2 \quad K = \frac{1}{2}mv^2$$

$$W = \tau\Delta\theta \quad W = F_{\parallel}d$$

$$L = I\omega \quad p = mv$$

$$\Delta L = \tau\Delta t \quad \Delta p = F\Delta t$$

The sheet also has  $L = rmv \sin \theta$  for an object moving past a point,  $v = r\omega$  and  $\Delta x_{cm} = r\Delta\theta$  for rolling,  $U_G = -Gm_1m_2/r$ , and Unit 5's  $\tau = r_{\perp}F$ . A disk's or hoop's  $I$  is given in the question.

**Not on the sheet, so know these by heart:**

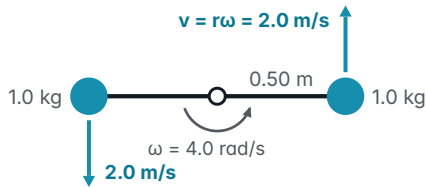
- An object that moves and spins has  $K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$ .
- The sheet's  $v = r\omega$  fits a rolling object only if it doesn't slip; then  $v$  is the speed of its center.
- With no outside torque on a system,  $L_i = L_f$ .
- For the same  $L$ ,  $K = L^2/2I$ , so the smaller  $I$  has more  $K$ .
- Circular orbit:  $v = \sqrt{GM/r}$ . Escape speed:  $v = \sqrt{2GM/r}$ .
- $\omega$  goes in rad/s: one revolution is  $2\pi$  rad.

## 6.1 Rotational kinetic energy

A spinning object has kinetic energy even when its center never moves

Every piece of a spinning object moves at  $v = r\omega$  and has kinetic energy  $\frac{1}{2}mv^2$ . All the pieces share the same  $\omega$ , so  $\omega^2$  comes out of the sum, and the sum that is left,  $\sum m_i r_i^2$ , is  $I$ :

$$K = \sum \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} (\sum m_i r_i^2) \omega^2 = \frac{1}{2} I \omega^2$$



The center of this dumbbell stays still, but each ball moves at 2.0 m/s, so  $K = 2 \times \frac{1}{2} (1.0) (2.0)^2 = 4.0 \text{ J}$ .  $\frac{1}{2} I \omega^2$  gives the same 4.0 J, with  $I = 2(1.0)(0.50)^2 = 0.50 \text{ kg}\cdot\text{m}^2$ .

**Quick check**

A hoop and a disk (same  $M$ ,  $R$ ) spin at the same  $\omega$ . Which has more  $K$ ?

**Answer:** the hoop. Its  $I$  is  $MR^2$ , twice the disk's  $\frac{1}{2}MR^2$ , so at the same  $\omega$  it has twice the  $K$ .

**On the equation sheet**

$$K = \frac{1}{2} I \omega^2$$

For an object turning about a fixed axle or pivot, use  $I$  about that axle or pivot.

**Not on the equation sheet: an object that moves and spins**

$$K_{\text{tot}} = \frac{1}{2} M v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega^2$$

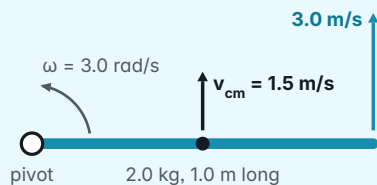
Use the rotational inertia about the center of mass here. If the object turns about a **fixed pivot** (a door, a swinging rod),  $\frac{1}{2} I_{\text{pivot}} \omega^2$  is already its whole kinetic energy, so don't add  $\frac{1}{2} M v_{\text{cm}}^2$  on top. The rod example below shows both ways.

**Worked example**

A 2.0 kg solid ball ( $I_{\text{cm}} = \frac{2}{5} MR^2$ ) rolls without slipping at 3.0 m/s. Find its total kinetic energy.

- Translational (moving along):  
 $\frac{1}{2} M v^2 = \frac{1}{2} (2.0) (3.0)^2 = 9.0 \text{ J}$
- Rotational (spinning), with  $\omega = v/R$ :  
 $\frac{1}{2} (\frac{2}{5} MR^2) \frac{v^2}{R^2} = \frac{1}{5} M v^2 = 3.6 \text{ J}$

**$K_{\text{tot}} = 12.6 \text{ J}$ ;  $R$  cancels, so any size of ball gives the same answer**

**Kinetic energy of a rod swinging about one end**

A 2.0 kg rod, 1.0 m long, swings about one end at 3.0 rad/s. The question gives its rotational inertia about the end,  $\frac{1}{3} ML^2$ , and about its center,  $\frac{1}{12} ML^2$ . Its center of mass moves at  $v_{\text{cm}} = \omega(L/2) = 1.5 \text{ m/s}$ .

**Turning about the pivot**

$$I_{\text{end}} = \frac{1}{3} (2.0) (1.0)^2 = 0.667 \text{ kg}\cdot\text{m}^2$$

$$K = \frac{1}{2} (0.667) (3.0)^2 = 3.0 \text{ J}$$

**Moving plus spinning about the center**

$$I_{\text{cm}} = \frac{1}{12} (2.0) (1.0)^2 = 0.167 \text{ kg}\cdot\text{m}^2$$

$$K = \frac{1}{2} (2.0) (1.5)^2 + \frac{1}{2} (0.167) (3.0)^2 = 2.25 + 0.75 = 3.0 \text{ J}$$

Both ways give 3.0 J. The mistake is to mix them:  $\frac{1}{2} I_{\text{end}} \omega^2 + \frac{1}{2} M v_{\text{cm}}^2$  gives 5.25 J, because the motion of the center is already inside  $I_{\text{end}}$ .

**Rotational inertias you will meet**

They are not on the equation sheet: when you need one, the question gives it. You should still know which is bigger. For the same mass and size, the farther the mass sits from the axis, the bigger  $I$  is.

| Shape, and the axis it turns about        | $I$                 | Where the mass sits                       |
|-------------------------------------------|---------------------|-------------------------------------------|
| Point mass, at distance $r$ from the axis | $mr^2$              | all at distance $r$                       |
| Hoop or thin ring, about its center       | $MR^2$              | all at the rim                            |
| Solid disk or cylinder, about its center  | $\frac{1}{2} MR^2$  | spread from the center to the rim         |
| Solid sphere, about its center            | $\frac{2}{5} MR^2$  | more of it near the center than in a disk |
| Thin rod, about its center                | $\frac{1}{12} ML^2$ | at most $L/2$ from the axis               |
| Thin rod, about one end                   | $\frac{1}{3} ML^2$  | up to $L$ from the axis                   |

**Worked example**

A flywheel with  $I = 2.0 \text{ kg}\cdot\text{m}^2$  turns at 300 rev/min. Find its kinetic energy.

- $\omega = 300 \times \frac{2\pi \text{ rad}}{60 \text{ s}} = 31.4 \text{ rad/s}$
- $K = \frac{1}{2} (2.0) (31.4)^2$

**$K \approx 990 \text{ J}$**

**Exam trap**

- Putting rev/min into  $\frac{1}{2} I \omega^2$ .** The formula only works with  $\omega$  in rad/s. For the flywheel, putting in 300 gives 90,000 J, about 91 times the right answer. Convert first: multiply by  $2\pi$  and divide by 60.
- Assuming more rotational inertia always means more kinetic energy.** That is true only at the same  $\omega$ . With the same angular momentum it is the other way round (the table below shows why).
- Saying  $K = 0$  because the center isn't moving.** Every piece of a spinning object moves, so it has kinetic energy. The dumbbell has 4.0 J while its center stays still.

| If this stays the same | and this doubles | Result                 |
|------------------------|------------------|------------------------|
| $I$                    | $\omega$         | $K \times 4$           |
| $\omega$               | $I$              | $K \times 2$           |
| $L$                    | $I$              | $K \times \frac{1}{2}$ |
| $K$                    | $I$              | $\omega \div \sqrt{2}$ |

The third row comes from putting  $\omega = L/I$  into  $\frac{1}{2} I \omega^2$ , which gives  $K = L^2/2I$ . With the same angular momentum, the object with more rotational inertia has *less* kinetic energy: with  $L = 4.0 \text{ kg}\cdot\text{m}^2/\text{s}$ ,  $I = 1.0 \text{ kg}\cdot\text{m}^2$  gives 8.0 J, but  $I = 2.0 \text{ kg}\cdot\text{m}^2$  gives 4.0 J.

**Worked example**

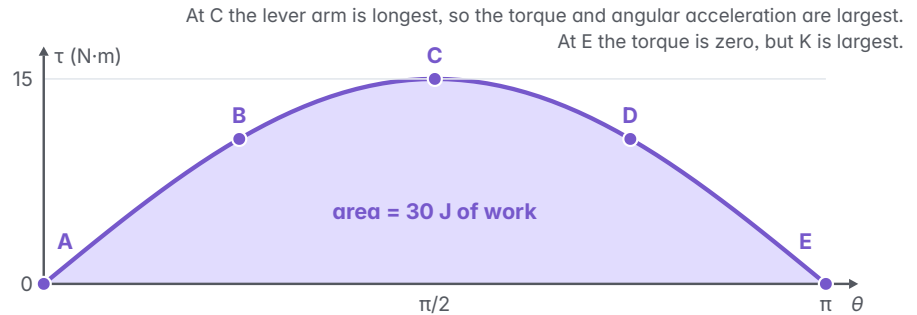
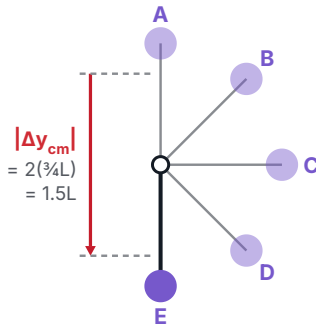
A brake exerts a steady 20 N·m against the spin of the flywheel above (990 J). How many turns does the flywheel make before it stops?

- The brake's work =  $\Delta K$ :  $-(20)\Delta\theta = -990 \text{ J}$
- $\Delta\theta = 990/20 = 49.5 \text{ rad}$ , and  $49.5/2\pi = 7.9$

**About 7.9 turns**

## 6.2 Torque and work

The work a torque does is the area under the torque-angle graph



A 1.0 kg rod, 1.0 m long, with a 1.0 kg sphere on its end, tips over from straight up at A and swings down to straight down at E. Together  $M = 2.0$  kg, and the center of mass sits halfway between the rod's middle and the sphere:  $\frac{3}{4}L = 0.75$  m from the axle. Gravity's torque is  $Mg(\frac{3}{4}L) \sin \theta$ , where  $\theta$  is the angle from straight up. The area under the graph, 30 J, equals  $Mg$  times the 1.5 m drop of the center of mass, and all of it becomes rotational kinetic energy.

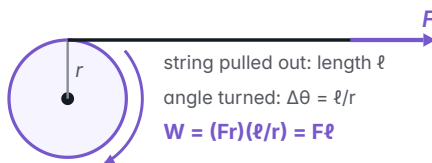
## On the equation sheet

$$W = \tau \Delta\theta \quad \text{and} \quad \Delta K = \sum W_i$$

$W = \tau \Delta\theta$  is for a constant torque, with  $\Delta\theta$  in radians. When the torque changes, the work is the area under the torque-angle graph.

**A torque does work only while the object turns.** Lean on a wrench when the bolt is stuck and you exert a large torque but do no work. A torque in the direction of the spin does positive work; one against it, such as friction in a bearing, takes energy out.

## A string pulled off a spool



The radius cancels, so the work is just the force times the length of string pulled out, whatever the size of the spool.

## Quick check

You push the rim of a merry-go-round, 2.0 m from its axle, with a steady 50 N along the rim while it turns half a revolution. How much work do you do?

**Answer:**  $\tau = (2.0)(50) = 100$  N·m and half a revolution is  $\pi$  rad, so  $W = 100\pi \approx 310$  J.

## Worked example

Two toys, X and Y, start at rest. A string wound on each is pulled with 8.0 N through 0.50 m.  $I_X = 0.020$  kg·m<sup>2</sup> and  $I_Y = \frac{1}{2}I_X$ . Find each  $\omega$ .

- $W = F\ell = (8.0)(0.50) = 4.0$  J for each toy
- $4.0 = \frac{1}{2}I\omega^2 \Rightarrow \omega = \sqrt{2W/I}$
- $\omega_X = \sqrt{8.0/0.020}$ ,  $\omega_Y = \sqrt{8.0/0.010}$

$$\omega_X = 20 \text{ rad/s and } \omega_Y = 28 \text{ rad/s}$$

## Writing a derivation for the toys

- Start from a law on the equation sheet:**  $\Delta K = \sum W_i$ , or  $W = \tau \Delta\theta$ .
- Put in the question's symbols** (force  $F_0$ , string length  $\ell_0$ ):  $F_0\ell_0 = \frac{1}{2}I_X\omega_X^2$ .
- Solve for what is asked:**  
 $\omega_X = \sqrt{2F_0\ell_0/I_X}$ .

Then say what it means:  $\omega$  is proportional to  $1/\sqrt{I}$ , so half the rotational inertia gives  $\sqrt{2} \approx 1.4$  times the angular speed.

## How to earn the point

The same force acts over the same distance, so the **work is the same** and both gain the **same kinetic energy**. Since  $K = \frac{1}{2}I\omega^2$ , the toy with **less rotational inertia** ends with the **greater angular speed**.

Writing only "because energy is conserved" explains nothing and doesn't earn the point.

## Worked example

The rod above swings from A to E. Find its gain in  $K$ .

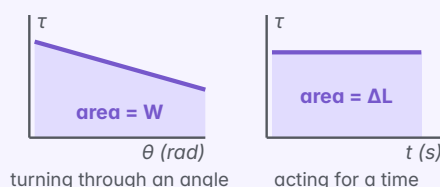
- $|\Delta y_{cm}| = 2(\frac{3}{4}L) = 1.5$  m
- $\Delta K = -\Delta U_g = Mg|\Delta y_{cm}| = (2.0)(10)(1.5)$

$$\Delta K = 30 \text{ J} = \frac{3}{2}MgL$$

## Exam trap

- Using the rod's length as the drop.** Gravity's work depends on how far the center of mass falls. Here it goes from  $\frac{3}{4}L$  above the axle to  $\frac{3}{4}L$  below it, a drop of  $2(\frac{3}{4}L) = 1.5$  m, not  $L = 1.0$  m.
- Saying force when you mean torque.** Gravity's force on the rod is the same at A, C and E, but its torque is not, because the lever arm changes. Write "the lever arm is largest at C", not "the force is largest at C".
- Losing track of the energy when Earth is not in the system.** Then the system has no gravitational potential energy. Gravity is an outside force that does positive work on the rod as its center falls, and that work becomes the rod's  $K$ .

## The same force over the same distance, or the same torque for the same time



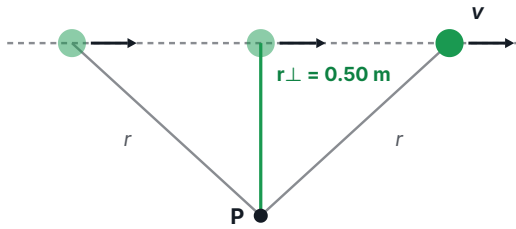
With the same  $K$ ,  $\omega = \sqrt{2K/I}$ ; with the same  $L$ ,  $\omega = L/I$  and  $K = L^2/2I$ . Either way the smaller  $I$  spins faster, and with the same  $L$  it also has more kinetic energy.

A torque acting while an object turns through an angle does **work**, which changes its kinetic energy. A torque acting for a time gives an **angular impulse**, which changes its angular momentum.

- The same force pulling the same length of string does the same work, so it gives the **same change in  $K$** .
- The same torque for the same time gives the same angular impulse, so the **same change in  $L$** .

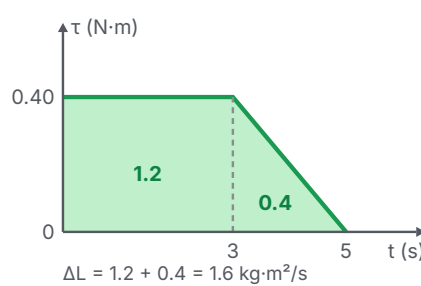
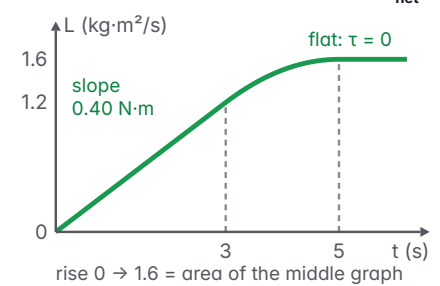
## 6.3 Angular momentum and angular impulse

Angular momentum changes only when an outside torque acts for a time

Straight-line motion still has  $L$  about  $P$ 

$$L = mvr_{\perp} = (0.20)(3.0)(0.50) = 0.30 \text{ kg}\cdot\text{m}^2/\text{s}$$

Left: a 0.20 kg puck coasting at 3.0 m/s has the same  $L$  about  $P$  at every point of its path, because its perpendicular distance from  $P$ ,  $r_{\perp} = r \sin \theta$ , never changes. About any point on its path,  $L = 0$ . Middle and right show the same wheel, so the area under the middle graph equals the rise of the right one.

Torque vs time: area =  $\Delta L$ Angular momentum vs time: slope =  $\tau_{\text{net}}$ 

## On the equation sheet

$L = I\omega$  rigid body, about its axis

$L = rmv \sin \theta$  a particle, about a point

$\Delta L = \tau \Delta t$  angular impulse

If the torque changes, use the average net torque in  $\Delta L = \tau_{\text{avg}} \Delta t$ . Angular momentum is measured in  $\text{kg}\cdot\text{m}^2/\text{s}$ , which is the same as  $\text{N}\cdot\text{m}\cdot\text{s}$ .

**A moving object's angular momentum depends on the point you measure it about.** In  $L = rmv \sin \theta$ ,  $\theta$  is the angle between  $r$  (from the point to the object) and  $v$ , and  $r \sin \theta = r_{\perp}$  is the perpendicular distance from the point to the object's line of motion. So  $L = mvr_{\perp}$ .

## Know by heart

- Pick one turning direction, say counterclockwise, as positive, and use it for every torque and every  $L$  in the problem.
- Angular impulse has the same sign as the torque that gives it.
- A force aimed straight through the axis has no lever arm, so it exerts no torque and can't change  $L$ .
- For a point mass going around a circle,  $I = mr^2$  and  $\omega = v/r$ , so  $L = mvr$ : the two formulas for  $L$  agree.
- $\Delta L = \tau \Delta t$  is Newton's second law for rotation,  $\tau = I\alpha$ , applied over a stretch of time.

## Quick check

A wheel's  $L$  falls in a straight line from 6.0 to 2.0  $\text{kg}\cdot\text{m}^2/\text{s}$  in 4.0 s. What is the net torque on it?

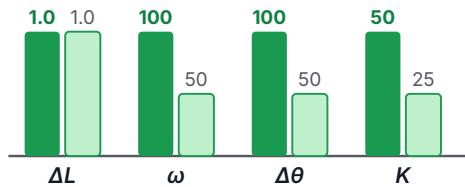
**Answer:** the slope,  $(2.0 - 6.0)/4.0 = -1.0 \text{ N}\cdot\text{m}$ . It is negative because it acts against the spin, as friction in a bearing would.

## Worked example

A disk ( $I = 0.010 \text{ kg}\cdot\text{m}^2$ ) and a hoop ( $I = 0.020 \text{ kg}\cdot\text{m}^2$ ) of the same mass and radius 0.10 m start at rest. Each string is pulled with 5.0 N for 2.0 s. Compare them.

- $\tau = rF = (0.10)(5.0) = 0.50 \text{ N}\cdot\text{m}$  on each
- $\Delta L = \tau \Delta t = 1.0 \text{ kg}\cdot\text{m}^2/\text{s}$  on each
- $\omega = L/I$ : disk 100, hoop 50 rad/s
- $K = L^2/2I$ : disk 50 J, hoop 25 J

**The same  $L$  for both, but the disk ends with twice the  $\omega$  and the  $K$**



Solid bars: the disk. Pale bars: the hoop. Each pair has its own scale:  $\Delta L$  in  $\text{kg}\cdot\text{m}^2/\text{s}$ ,  $\omega$  in rad/s,  $\Delta \theta$  in rad,  $K$  in J. The disk turns through twice the angle, so the same force does twice the work on it.

## How to learn the point

The **same torque** acts on each for the **same time**, so each gets the **same angular impulse** and the same  $\Delta L$ . The disk's mass sits closer to the axis, so its  $I$  is **half** the hoop's and  $\omega = L/I$  is **double**. With the same  $L$ ,  $K = L^2/2I$ , so half the  $I$  means **twice the  $K$** .

In the lab: the slope of  $L$  against  $t$  is the net torque

A string pulls a flywheel ( $I = 0.25 \text{ kg}\cdot\text{m}^2$ ) with a steady 4.0 N at a hub of radius 0.050 m. A photogate measures  $\omega = 0, 0.64, 1.28, 1.92, 2.56 \text{ rad/s}$  at  $t = 0, 1, 2, 3, 4 \text{ s}$ . Plot  $L = I\omega$  against  $t$ : the slope is the net torque, **0.16 N·m**. The string alone exerts  $rF = 0.20 \text{ N}\cdot\text{m}$ , so friction in the bearing must exert **-0.04 N·m**.

## Exam trap

- Calling it conservation of angular momentum.** When a string pulls each wheel, an outside torque acts, so neither keeps its angular momentum. Their changes are equal because the torque and the time are equal: 0.50 N·m for 2.0 s gives each 1.0  $\text{kg}\cdot\text{m}^2/\text{s}$ .
- Using force where you need torque.**  $\tau = r_{\perp}F$ , so the same 5.0 N pulled at a radius of 0.050 m instead of 0.10 m gives half the torque and half the  $\Delta L$ .
- Saying an object moving in a straight line has no angular momentum.** It has  $L = mvr_{\perp}$  about any point off its path.  $L$  is zero only about points on the line it moves along.

## Worked example

A torque on a wheel ( $I = 0.080 \text{ kg}\cdot\text{m}^2$ , at rest) follows the middle graph above. Find the wheel's final  $\omega$ .

- Area =  $(0.40)(3.0) + \frac{1}{2}(0.40)(2.0) = 1.6 \text{ N}\cdot\text{m}\cdot\text{s}$
- $\omega = \Delta L/I = 1.6/0.080$

$$\omega = 20 \text{ rad/s}$$

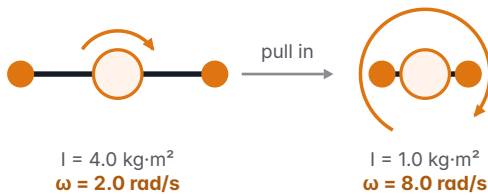
## One push can change both kinds of momentum

Two balls get the same push for the same short time. One push is aimed through the center; the other is aimed a little to the side.

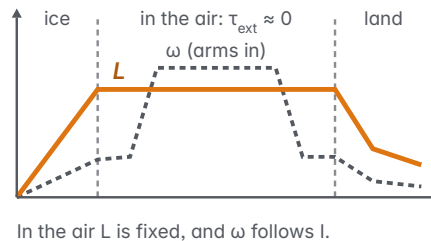
Both gain the **same linear momentum** ( $\vec{J} = \vec{F}_{\text{avg}} \Delta t$ ). Only the off-center push has a lever arm, so only that ball also starts to spin and gains **angular momentum**. Find the change in  $\vec{p}$  from the force and the change in  $L$  from the torque.

## 6.4 Conservation of angular momentum

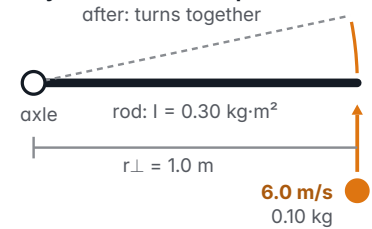
With no net outside torque, a system's total angular momentum stays constant

Arms pulled in:  $I$  falls, so  $\omega$  rises $L: 8.0 \rightarrow 8.0 \text{ kg}\cdot\text{m}^2/\text{s}$  $K: 8.0 \text{ J} \rightarrow 32 \text{ J}$  (the arms did work)

Left: pulling the weights in cuts  $I$  to a quarter, so  $\omega$  becomes four times as large, and so does the kinetic energy, because the arms do work pulling the weights in. Middle: the ice's torque builds up the skater's angular momentum; in the air nothing outside twists the skater, so  $L$  stays flat while  $\omega$  rises as the arms come in. Right: the axle pushes on the rod during the hit, so linear momentum is not conserved, but that push acts at the axle and has no torque about it, so  $L$  about the axle is conserved.

 $L$  and  $\omega$  during a spin jump

Clay sticks to a rod (top view)

 $L$  before =  $mvr_{\perp} = 0.60 \text{ kg}\cdot\text{m}^2/\text{s}$ after:  $(0.30 + 0.10)\omega$ , so  $\omega = 1.5 \text{ rad/s}$ 

Not on the equation sheet: when angular momentum is conserved

$$\tau_{\text{net,ext}} = 0 \Rightarrow L_i = L_f$$

If no net torque acts on a system from outside, its total angular momentum stays the same. Torques between its parts come in equal and opposite pairs, so they only pass  $L$  from one part to another.

When the outside torque is zero, or close to it

- On a frictionless axle, the axle's force acts at the axis, so it has no lever arm and no torque.
- In a brief collision, an outside torque changes  $L$  by only  $\tau\Delta t$ , which is small when  $\Delta t$  is short, unless that torque is large.
- In the air (a diver, a thrown baton), gravity acts at the center of mass, so it exerts no torque about the center of mass.

## Worked example

A person on a stool spins at  $2.0 \text{ rad/s}$ . Pulling in their arms cuts  $I$  from  $4.0$  to  $1.0 \text{ kg}\cdot\text{m}^2$ . Find  $\omega_f$  and the change in  $K$ .

- $I_i\omega_i = I_f\omega_f: (4.0)(2.0) = (1.0)\omega_f$
- $K_i = \frac{1}{2}(4.0)(2.0)^2 = 8.0 \text{ J}; K_f = \frac{1}{2}(1.0)(8.0)^2$

$\omega_f = 8.0 \text{ rad/s}; K$  rises from  $8.0 \text{ J}$  to  $32 \text{ J}$ , from the work the arms do

## Quick check

A child walks from the rim of a freely turning merry-go-round to its center. What happens to  $\omega$  and to  $K$ ?

**Answer:**  $I$  falls, so  $\omega$  rises.  $K = L^2/2I$  also rises; the extra energy comes from the work the child does walking inward.

## Worked example

The clay above ( $0.10 \text{ kg}$ ,  $6.0 \text{ m/s}$ ,  $1.0 \text{ m}$  from the axle) sticks to the end of the rod ( $I = 0.30 \text{ kg}\cdot\text{m}^2$ ). Find  $\omega$  and the  $K$  lost.

- $L_i = rmv \sin 90^\circ = (1.0)(0.10)(6.0) = 0.60 \text{ kg}\cdot\text{m}^2/\text{s}$
- $I_f = 0.30 + (0.10)(1.0)^2 = 0.40 \text{ kg}\cdot\text{m}^2$
- $\omega = 0.60/0.40 = 1.5 \text{ rad/s}$
- $K: \frac{1}{2}(0.10)(6.0)^2 = 1.8 \text{ J} \rightarrow \frac{1}{2}(0.40)(1.5)^2 = 0.45 \text{ J}$

$\omega = 1.5 \text{ rad/s}$ , and  $1.35 \text{ J}$  (75% of the  $K$ ) becomes internal energy

## If the clay bounces back instead

Suppose the same clay bounces straight back at  $2.0 \text{ m/s}$ . Its angular momentum about the axle goes from  $+0.60$  to  $-0.20 \text{ kg}\cdot\text{m}^2/\text{s}$ , so the rod gains  $0.80 \text{ kg}\cdot\text{m}^2/\text{s}$  and turns at  $0.80/0.30 = 2.7 \text{ rad/s}$ , almost twice as fast as when the clay sticks. An object that bounces back hands over more momentum than one that stops, just as in the collisions of Unit 4.

## Setting up a conservation problem

- Choose the system so the torques that matter are inside it: the clay and the rod, not the rod alone.
- Add up  $L$  before:  $I\omega$  for each spinning object and  $mvr_{\perp}$  for each object moving in a line, with signs.
- Set that equal to  $L$  after, using the new rotational inertia (include anything that stuck on).
- Work out kinetic energy separately if asked. It is not conserved just because  $L$  is.

Justifying that  $L$  is conserved

Take the clay and rod as the system. The outside forces are gravity, which pulls **parallel to the vertical axle**, and the axle's push, which acts **at the axle**. Neither exerts a torque about the axle, so the net outside torque is zero and the system's **angular momentum is constant**. The clay's  $L$  before equals the clay-and-rod  $L$  after.

## Exam trap

- Assuming kinetic energy is conserved because angular momentum is.  $K = L^2/2I$  changes whenever  $I$  does: the person on the stool keeps  $L = 8.0 \text{ kg}\cdot\text{m}^2/\text{s}$  but goes from  $8.0 \text{ J}$  to  $32 \text{ J}$ .
- Using the rod's old rotational inertia after the clay sticks. The clay turns with the rod, so  $I_f = I_{\text{rod}} + mr^2 = 0.40 \text{ kg}\cdot\text{m}^2$ .
- Conserving linear momentum when there is a pivot. The axle pushes on the rod during the hit, so linear momentum changes. Use angular momentum about the axle.
- Using energy conservation when the clay sticks.  $\frac{1}{2}mv^2 = \frac{1}{2}I\omega^2$  gives the wrong  $\omega$ , because 75% of the clay's  $K$  becomes internal energy.

## Worked example

A disk ( $I = 0.020 \text{ kg}\cdot\text{m}^2$ ) spinning at  $30 \text{ rad/s}$  drops onto a disk at rest ( $I = 0.010 \text{ kg}\cdot\text{m}^2$ ) on the same axle, and they turn together. Find  $\omega$ .

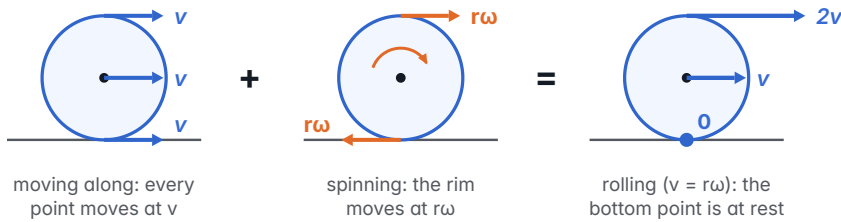
- $(0.020)(30) = (0.020 + 0.010)\omega$
- $K: 9.0 \text{ J} \rightarrow \frac{1}{2}(0.030)(20)^2 = 6.0 \text{ J}$

$\omega = 20 \text{ rad/s}$ ; a third of the  $K$  is lost

## 6.5 Rolling

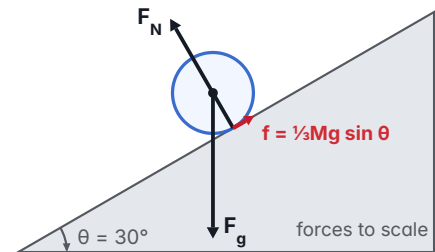
Rolling without slipping is moving along plus spinning, tied together by  $v_{\text{cm}} = r\omega$ 

## Moving along plus spinning makes rolling



Left: the two motions cancel at the bottom only when  $v = r\omega$ , which is what rolling without slipping means. Right: on a ramp, static friction acts **up the slope at the contact point**. It has a lever arm about the center, so its torque is what makes the disk spin faster. Gravity acts at the center and the normal force points through it, so neither has a lever arm about the center.

## Forces on a rolling disk



## On the equation sheet

$$\Delta x_{\text{cm}} = r\Delta\theta \quad v = r\omega \quad a_T = r\alpha$$

For a rolling object,  $v$  and  $a_T$  (acceleration along the path) are its center's.

## Know by heart

- When an object rolls without slipping along a straight surface,  $v_{\text{cm}} = r\omega$  and  $a_{\text{cm}} = r\alpha$ . If it slips, neither holds.
- $K_{\text{tot}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$
- Static friction on a rolling object **does no work**, because the point where it acts isn't sliding. On a ramp it is **not zero**, though: it supplies the torque.

## Speed at the bottom of a ramp

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I_{\text{cm}}\left(\frac{v}{R}\right)^2$$

$$v = \sqrt{\frac{2gh}{1 + I_{\text{cm}}/MR^2}}$$

It isn't on the equation sheet, so derive it from the first line.  $M$  and  $R$  cancel, so a big disk and a small disk tie.

## Worked example

A solid disk ( $I_{\text{cm}} = \frac{1}{2}MR^2$ , given) rolls from rest down a 1.5 m drop. Find  $v$  at the bottom.

- $v^2 = \frac{2(10)(1.5)}{1 + \frac{1}{2}} = 20 \text{ m}^2/\text{s}^2$
- $K_{\text{rot}}/K_{\text{trans}} = I_{\text{cm}}/MR^2 = \frac{1}{2}$

$v = 4.5 \text{ m/s}$ , and one third of its  $K$  is rotational

## Quick check

A hoop rolls without slipping at 2.0 m/s. What share of its  $K$  is rotational? How fast does its top move?

**Answer:** half, since  $I_{\text{cm}}/MR^2 = 1$ ; the top moves at  $2v = 4.0 \text{ m/s}$ .

## Worked example

Derive the acceleration of a disk ( $I = \frac{1}{2}MR^2$ ) rolling without slipping down a ramp at angle  $\theta$ .

- Along the ramp:  $Mg \sin \theta - f = Ma$
- About the center:  $fR = (\frac{1}{2}MR^2)\alpha$
- No slipping, so  $\alpha = a/R$ :  $f = \frac{1}{2}Ma$
- $Mg \sin \theta = \frac{3}{2}Ma$

$a = \frac{2}{3}g \sin \theta$ . For 3.0 kg on a  $30^\circ$  ramp,  $a = 3.3 \text{ m/s}^2$  and  $f = 5.0 \text{ N}$

| Object                      | $I_{\text{cm}}$   | Acceleration               | Share of $K$ that is rotational |
|-----------------------------|-------------------|----------------------------|---------------------------------|
| Sliding block (no friction) | (no spin)         | $g \sin \theta$            | 0                               |
| Solid sphere                | $\frac{2}{5}MR^2$ | $\frac{5}{7}g \sin \theta$ | 2/7                             |
| Solid disk                  | $\frac{1}{2}MR^2$ | $\frac{2}{3}g \sin \theta$ | 1/3                             |
| Hoop                        | $MR^2$            | $\frac{1}{2}g \sin \theta$ | 1/2                             |

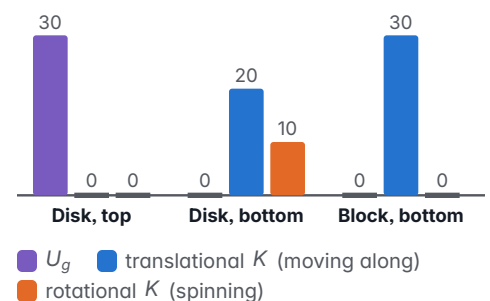
In general  $a = g \sin \theta / (1 + I_{\text{cm}}/MR^2)$ : the farther out the mass sits, the bigger  $I_{\text{cm}}/MR^2$  and the slower the object.

## Which reaches the bottom first, a disk or a block?

**Using forces:** nothing opposes the block's slide, but static friction acts up the ramp on the disk, so the disk has the **smaller acceleration**. From rest over the same distance, the **block arrives first**.

**Using energy:** both lose the same  $U_g$  per kilogram, but the disk must put part of it into **rotation**, leaving less translational  $K$  and a lower speed at every point.

## The energy as bar charts, in joules



A 2.0 kg disk rolls from rest down a 1.5 m drop, losing  $Mgh = 30 \text{ J}$  of  $U_g$ . It ends with 20 J from moving along and 10 J from spinning, so its speed comes from only 20 J:  $\frac{1}{2}(2.0)v^2 = 20$  gives  $v = 4.5 \text{ m/s}$ . A frictionless block puts all 30 J into moving along, so it is faster: 5.5 m/s.

## Quick check

A disk and a hoop roll from rest down the same ramp. Find  $v_{\text{disk}}/v_{\text{hoop}}$  at the bottom.

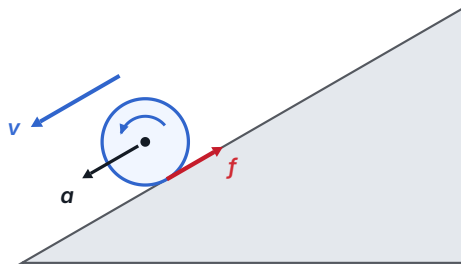
**Answer:**  $\sqrt{(1+1)/(1+\frac{1}{2})} = \sqrt{4/3} \approx 1.15$ .

## Exam trap

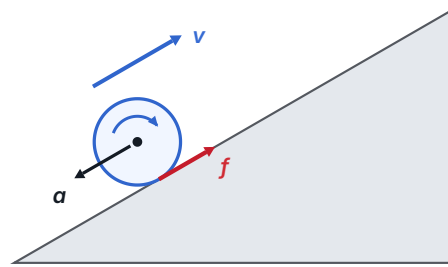
- Answering "because energy is conserved". Say where the energy goes instead: the disk's lost potential energy becomes both translational and rotational kinetic energy, so it has less translational  $K$  than a sliding block.
- Thinking the heavier or bigger ball wins.  $M$  and  $R$  cancel in  $v = \sqrt{2gh/(1 + I_{\text{cm}}/MR^2)}$ , so two solid balls of any size tie. Only the shape matters.

## 6.5 More rolling problems

On a ramp, static friction on a rolling object points up the slope



**Rolling down,**  
speeding up: the  
torque from friction  
speeds up the spin



**Rolling up,**  
slowing down: the  
torque from friction  
slows the spin

— velocity  $v$  of the center — acceleration  $a$  of the center, down the slope both times — static friction  $f$

## Rolling up a ramp

Going up, the object slows down and its spin has to slow down with it. Gravity slows the center, and static friction, **still pointing up the slope**, slows the spin. Both kinds of kinetic energy turn into height.

## Worked example

A hoop rolls at 3.0 m/s onto an upward ramp and rolls without slipping until it stops. How high does it climb? Compare a frictionless block at 3.0 m/s.

1 Hoop:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(MR^2)\frac{v^2}{R^2} = Mv^2,$$

so  $h = v^2/g$

2 Block:  $Mgh = \frac{1}{2}Mv^2$ , so  $h = v^2/2g$

**Hoop 0.90 m, block 0.45 m: the hoop's spin energy also turns into height**

## Worked example

A bike wheel of radius 0.35 m rolls 22 m without slipping. How many turns does it make?

1  $\Delta x_{cm} = r\Delta\theta$ , so  $\Delta\theta = 22/0.35 = 63 \text{ rad}$

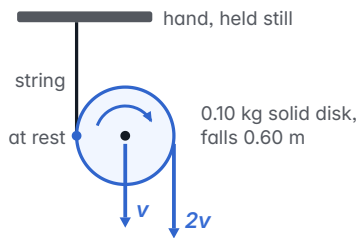
2 One turn is  $2\pi \text{ rad}$ :  $63/2\pi = 10$

**10 turns: each turn rolls it one circumference,  $2\pi r = 2.2 \text{ m}$**

## Exam trap

- Thinking friction takes energy from a rolling object. When it rolls without slipping, static friction does no work. It only shifts some of the energy from moving along into spinning.
- Using  $v = r\omega$  while the ball still skids. It holds only once the slipping stops; while a bowling ball skids,  $v_{cm}$  is bigger than  $r\omega$ .

## A yo-yo unwinding from its string



The yo-yo turns about the point where the string leaves it. That point is at rest, so the string's tension does no work, just like friction on a rolling ball.

## Worked example

The yo-yo is let go from rest. Find its speed after it falls 0.60 m. Its  $I = \frac{1}{2}mr^2$ .

$$1 \quad mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\frac{v^2}{r^2} = \frac{3}{4}mv^2$$

$$2 \quad v = \sqrt{4gh/3} = \sqrt{4(10)(0.60)/3} = \sqrt{8.0}$$

**$v = 2.8 \text{ m/s}$ , slower than the 3.5 m/s of a ball dropped 0.60 m**

## Worked example

A block ( $m = 2.0 \text{ kg}$ ) hangs from a string wrapped around a pulley, a disk ( $M = 4.0 \text{ kg}$ ,  $I = \frac{1}{2}MR^2$ ) on a frictionless axle. The string doesn't slip. Find the block's speed after it falls 1.0 m from rest.

$$1 \quad mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{v^2}{R^2}$$

$$2 \quad (2.0)(10)(1.0) = \frac{1}{2}(2.0 + 2.0)v^2, \text{ so } v^2 = 10$$

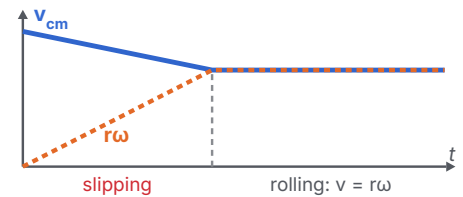
**$v = 3.2 \text{ m/s}$ ; with a pulley of no mass it would be 4.5 m/s**

## Quick check

A solid ball and a hollow ball, same mass and radius, race down a ramp. Which wins?

**Answer:** the solid ball. Less of its mass is far from the center, so  $I_{cm}/MR^2$  is smaller and less of its energy goes into spinning.

## A ball that slides before it rolls



A bowling ball leaves the hand sliding, with no spin. Kinetic friction pushes back on it, which slows  $v_{cm}$ , and the torque from that friction speeds up  $\omega$ , until  $v = r\omega$ . While it slips, friction does negative work and mechanical energy is lost. The exam asks you to explain this in words.

## Three ways down the same ramp

The wheel starts from rest each time and drops the same height. Its energy at the bottom:

| Case                                     | Moving along, $K_{trans}$ | Spinning, $K_{rot}$ | Mechanical energy left |
|------------------------------------------|---------------------------|---------------------|------------------------|
| 1. Rolls without slipping                | least                     | most                | all of it              |
| 2. Rolls and slips (too little friction) | middle                    | middle              | less                   |
| 3. Slides, no friction                   | most                      | 0                   | all of it              |

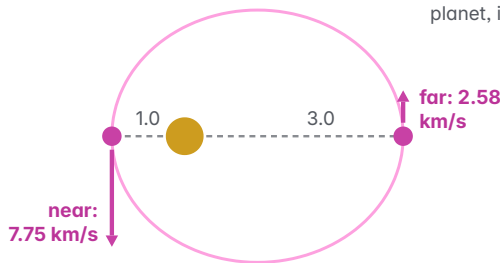
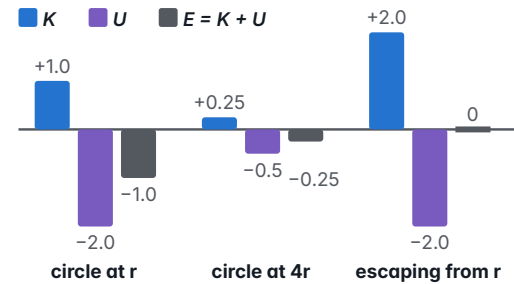
## Justifying the ranking

With no friction, nothing makes the wheel spin, so all the energy it loses goes into moving along: case 3 has the most  $K_{trans}$ . When it rolls without slipping, static friction does no work but makes the wheel spin, so the same energy is shared with rotation: case 1 has the least. When it slips, friction is weaker than rolling needs, so the wheel spins less and moves faster than in case 1, but slower than in case 3. Kinetic friction also turns some energy into internal energy, so case 2 keeps the least total energy.

## 6.6 Motion of orbiting satellites

Gravity is the only force, so the energy and the angular momentum stay fixed

## One orbit, to scale

distances from the planet, in  $10^7$  mEnergy of a 500 kg satellite ( $10^{10}$  J)

$$U = -GMm/r$$



always negative, rises toward 0

Left: around a planet with  $GM = 4.0 \times 10^{14} \text{ m}^3/\text{s}^2$ , a satellite on an ellipse moves fastest when it is nearest the planet and slowest when it is farthest away. At those two points its velocity is perpendicular to  $\vec{r}$ , so  $L = rmv$  and  $r_{\text{near}}v_{\text{near}} = r_{\text{far}}v_{\text{far}}$ . Middle: energies at  $r = 1.0 \times 10^7$  m, at  $4r$ , and for a launch from  $r$  that just escapes. In a circular orbit  $U = -2K$  and  $E = -K$ , so a bigger circle means a slower satellite but more total energy (less negative). To escape,  $E$  must reach 0.

## On the equation sheet

$$|\vec{F}_g| = G \frac{m_1 m_2}{r^2} \quad U_G = -\frac{Gm_1 m_2}{r}$$

$$a_c = \frac{v^2}{r} \quad K = \frac{1}{2}mv^2$$

$$L = rmv \sin \theta \quad C = 2\pi r$$

The sheet also gives  $G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$ , and  $\Delta U_g = mg\Delta y$  for heights near the ground, but none of the orbit formulas below.

## Deriving a circular orbit

Gravity alone supplies the centripetal force:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$$

One lap of length  $2\pi r$  at that speed:

$$T = \frac{2\pi r}{v} = 2\pi \sqrt{\frac{r^3}{GM}}, \text{ so } T^2 = \frac{4\pi^2}{GM} r^3$$

Energy, using  $mv^2 = GMm/r$ :

$$K = \frac{GMm}{2r}, \quad E = K + U_G = -\frac{GMm}{2r}$$

The satellite's mass cancels, so a pebble and a space station at the same  $r$  keep pace.

## Not on the equation sheet: escape speed

The slowest launch that escapes just reaches infinity with zero speed, so  $E = 0$ :

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = 0 \Rightarrow v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$$

That is  $\sqrt{2}$  times the circular-orbit speed at that  $r$ , in any direction that misses the planet.

## Quick check

On an elliptical orbit, where is  $K$  largest? Where is  $U_G$  largest?

**Answer:**  $K$  at the nearest point.  $U_G$  is largest (least negative) at the farthest point. Their sum is the same everywhere.

## Worked example

A planet has  $GM = 4.0 \times 10^{14} \text{ m}^3/\text{s}^2$ . Find the speed and period of a circular orbit of radius  $1.0 \times 10^7$  m, and the escape speed from there.

$$1 \quad v = \sqrt{(4.0 \times 10^{14})/(1.0 \times 10^7)} = 6.3 \times 10^3 \text{ m/s}$$

$$2 \quad T = 2\pi(1.0 \times 10^7)/(6.32 \times 10^3) = 9.9 \times 10^3 \text{ s}$$

$$3 \quad v_{\text{esc}} = \sqrt{2}(6.32 \times 10^3) = 8.9 \times 10^3 \text{ m/s}$$

$$v = 6.3 \text{ km/s}, T = 2.8 \text{ hours}, v_{\text{esc}} = 8.9 \text{ km/s}$$

## Worked example

The ellipse above:  $v_{\text{near}} = 7.75 \text{ km/s}$  at  $r_{\text{near}} = 1.0 \times 10^7$  m. Find  $v_{\text{far}}$  at  $3.0 \times 10^7$  m.

$$1 \quad L \text{ is constant and } \sin \theta = 1 \text{ at both points, so } r_{\text{near}}v_{\text{near}} = r_{\text{far}}v_{\text{far}}$$

$$2 \quad v_{\text{far}} = 7.75 \times \frac{1.0}{3.0}$$

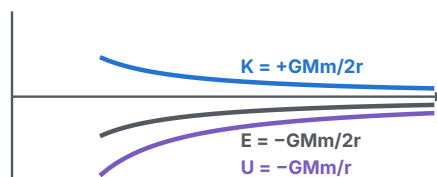
$$3 \quad \text{Check with the energy per kilogram, } \frac{1}{2}v^2 - GM/r. \text{ Near: } 3.0 \times 10^7 - 4.0 \times 10^7 = -1.0 \times 10^7 \text{ J/kg. Far: } 3.3 \times 10^6 - 1.33 \times 10^7 = -1.0 \times 10^7 \text{ J/kg.}$$

$$v_{\text{far}} = 2.58 \text{ km/s}$$

## Explaining why it speeds up near the planet

Gravity does **positive work** as the satellite moves closer, so  $U_G$  falls and  $K$  rises.  $L$  is constant all the way around, and at the near and far points, where  $\vec{v} \perp \vec{r}$ ,  $L = rmv$ . So the **smaller**  $r$  goes with the **larger**  $v$ .

| Circular orbit | 4 times the radius                 | twice the satellite's mass |
|----------------|------------------------------------|----------------------------|
| Speed          | $\times \frac{1}{2}$               | same                       |
| Period         | $\times 8$                         | same                       |
| Kinetic energy | $\times \frac{1}{4}$               | $\times 2$                 |
| Total energy   | $\times \frac{1}{4}$ , closer to 0 | $\times 2$                 |

Energy against  $r$  for circular orbitsEntering numbers like  $6.67 \times 10^{-11}$ 

**TI-84 Plus CE.** Type 6.67, then  $\boxed{2\text{nd}} \boxed{,}$  (EE), then  $\boxed{(-)}$  11. The screen shows 6.67E-11. Use  $\boxed{(-)}$  for a negative sign, never the subtraction key.

**TI-Nspire CX.** Type 6.67  $\boxed{\times 10}$   $\boxed{^{\wedge}}$   $\boxed{(-)}$  11. Use  $\boxed{(-)}$  for a negative sign, never the subtraction key.

**Desmos.** Type  $6.67 \times 10^{-11}$ .

## Exam trap

- Using the height above the surface as  $r$ .  $r$  is measured from the planet's center: 400 km above a planet of radius 6,400 km is  $r = 6,800$  km.
- Giving  $U_G$  a positive value.  $U_G = -Gm_1m_2/r$  is negative at every distance and rises toward zero as  $r$  grows, so a farther satellite has more (less negative)  $U_G$ .
- Thinking a higher circular orbit is faster.  $v = \sqrt{GM/r}$ , so at 4 times the radius the satellite moves at half the speed, even though its total energy is higher.
- Treating  $rv$  as constant at every point of the ellipse. It holds only at the nearest and farthest points, where  $\vec{v}$  is perpendicular to  $\vec{r}$ . Elsewhere  $L = rmv \sin \theta$ , so keep the  $\sin \theta$ .

## Which idea fits the question

What the question keeps the same tells you which idea to use

**It spins, and the question asks about energy**

Use  $K = \frac{1}{2}I\omega^2$ , adding  $\frac{1}{2}Mv_{\text{cm}}^2$  if the center moves too. About a fixed pivot,  $\frac{1}{2}I_{\text{pivot}}\omega^2$  alone is the whole  $K$ .

Put  $\omega$  in rad/s, never rev/min. (6.1)

**A force acts through a distance or an angle**

Use work and energy:  $W = \tau\Delta\theta$ , or  $F\ell$  for a string pulled off a spool, then  $\Delta K = \sum W_i$ .

The same force over the same distance gives the same  $\Delta K$ . (6.2)

**A torque acts for a time**

Use angular impulse:  $\Delta L = \tau\Delta t$ , or the area under the torque-time graph.

The same torque for the same time gives the same  $\Delta L$ . (6.3)

**No outside torque acts on the system**

When part of the system moves in or out, or two objects stick together, set  $L_i = L_f$  about one axis, using the new  $I$ .

Kinetic energy changes: it rises when parts pull in and falls when things stick. (6.4)

**It rolls without slipping**

Use  $v_{\text{cm}} = r\omega$ , then energy with both kinetic energy terms, or  $\sum F = Ma$  together with  $\sum \tau = I\alpha$ .

Static friction exerts a torque but does no work. (6.5)

**Gravity is the only force**

Total energy and  $L$  stay constant. For a circular orbit, set  $Gm_1m_2/r^2 = mv^2/r$ .

Measure  $r$  from the center;  $U_G$  is always negative. (6.6)

⊙ **A free-response question, answered part by part**

Wheels A and B have the same mass and radius, but B's mass sits near its rim, so  $I_B = 2I_A$ . Each starts at rest on a frictionless axle, and a string wrapped around each is pulled with the same constant force  $F_0$ .

| The question asks                                                                                                                | An answer that earns credit                                                                                                                                                                             | What graders look for                                                                                                  |
|----------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------|
| A. Both strings are pulled through the same length $\ell_0$ . Which wheel ends with more kinetic energy, and which spins faster? | Their kinetic energies are equal: the same force over the same length does the same work, $F_0\ell_0$ , on each. Since $K = \frac{1}{2}I\omega^2$ , A, with less rotational inertia, spins faster.      | Say what is the same (the work) and why, then use it. "Energy is conserved" on its own earns nothing.                  |
| B. Derive $\omega_B$ in terms of $F_0$ , $\ell_0$ and $I_A$ .                                                                    | $F_0\ell_0 = \frac{1}{2}I_B\omega_B^2 = \frac{1}{2}(2I_A)\omega_B^2$ , so $\omega_B = \sqrt{F_0\ell_0/I_A}$ .                                                                                           | Start from a law on the equation sheet ( $\Delta K = \sum W_i$ ) and end in the symbols asked for, with no $I_B$ left. |
| C. Is your expression consistent with part A?                                                                                    | Yes. The same steps give $\omega_A = \sqrt{2F_0\ell_0/I_A}$ , which is $\sqrt{2}$ times $\omega_B$ : the wheel with more rotational inertia spins slower.                                               | Point to how $\omega$ depends on $I$ in your expression.                                                               |
| D. Instead, both strings are pulled for the same time. Which wheel gains more angular momentum? More kinetic energy?             | They gain the same angular momentum: the same force at the same radius gives the same torque, and for the same time, the same angular impulse. $K = L^2/2I$ , so A gains twice the kinetic energy of B. | Don't call it conservation of angular momentum: an outside torque acts on each wheel.                                  |

**Other prompts you will see, and what to write**

| The prompt                                                | What to write                                                                                                                                                                                                                                  |
|-----------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| "In which position is the angular acceleration greatest?" | Where the <b>torque</b> is greatest: there the force is perpendicular to the rod, so the lever arm is longest, and $\alpha = \tau/I$ . (6.2)                                                                                                   |
| "Derive the change in $K$ as the rod swings down."        | Start from $\Delta K = -\Delta U_g$ and use how far the <b>center of mass</b> drops, not the rod's length. (6.2)                                                                                                                               |
| "Which reaches the bottom first?"                         | Use forces (friction up the slope lowers the rolling object's acceleration) or energy (part of the lost $U_g$ goes into spinning). Then say that from rest over the same distance, the one with the larger acceleration takes less time. (6.5) |
| "Compare the satellite's speed at two points."            | Use energy with $U_G = -Gm_1m_2/r$ , or $r_{\text{near}}v_{\text{near}} = r_{\text{far}}v_{\text{far}}$ at the nearest and farthest points: closer means faster. (6.6)                                                                         |
| "Sketch $L$ against $t$ ."                                | The slope of $L$ is the net torque: a straight line while a steady torque acts, flat when none does. (6.3)                                                                                                                                     |

**What happens to  $L$  and  $K$  in each event**

| What happens                                  | Angular momentum               | Kinetic energy  |
|-----------------------------------------------|--------------------------------|-----------------|
| A skater pulls in their arms                  | <b>constant</b>                | rises           |
| A disk drops onto a spinning disk             | <b>constant</b>                | falls           |
| Clay sticks to a rod on an axle               | <b>constant</b> about the axle | falls           |
| A child walks inward on a free merry-go-round | <b>constant</b>                | rises           |
| A satellite goes around an ellipse            | <b>constant</b>                | trades with $U$ |
| A hanging block turns a pulley as it falls    | grows                          | rises           |

In the last row the block's weight exerts an outside torque on the pulley-and-block system, so  $L$  grows. Linear momentum is not conserved when the clay hits (the axle pushes on the rod) or around the ellipse (gravity keeps pulling), even though  $L$  is.

**Choosing the system**

The Moon's pull on the tides gradually slows Earth's spin, so Earth on its own loses angular momentum. Take Earth and the Moon together as the system and that torque is internal: the  $L$  Earth loses goes into the Moon's orbit, which slowly widens.

| Quantity                | Units that are easy to mix up                                         |
|-------------------------|-----------------------------------------------------------------------|
| Torque $\tau$           | N·m. Never write it as J, even though a joule is also a newton-meter. |
| Work, $K$ and $U$       | J                                                                     |
| $L$ and angular impulse | kg·m <sup>2</sup> /s, which is the same as N·m·s                      |
| $\omega$ and $\alpha$   | rad/s and rad/s <sup>2</sup>                                          |