

Unit 7 Oscillations

AP Physics 1

Restoring force · Period and frequency · Position, velocity and acceleration · Energy

Unit 1 10–15%
Kinematics

Unit 2 18–23%
Force and dynamics

Unit 3 18–23%
Work, energy, power

Unit 4 10–15%
Linear momentum

Unit 5 10–15%
Torque and rotation

Unit 6 5–8%
Rotating systems

Unit 7 5–8%
Oscillations

Unit 8 10–15%
Fluids

Unit 7 is 5–8% of the multiple-choice section. On the free-response section, the College Board's sample question 2, which asks for energy bars, graphs and a force diagram that agree, is about a block on a spring.

7.1 Defining simple harmonic motion

7.2 Frequency and period of SHM

7.3 Representing and analyzing SHM

7.4 Energy of simple harmonic oscillators

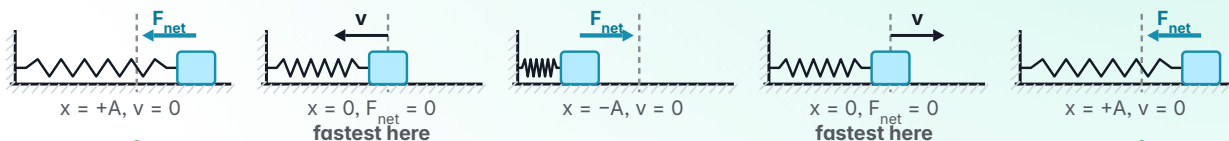
One cycle of a 0.50 kg cart on a 200 N/m spring, on a frictionless track, pulled 0.10 m to the right and let go.

7.2 Period

One period, $T = 2\pi\sqrt{m/k} = 0.314$ s, ends back at the same place, moving the same way

7.1

Restoring force toward $x = 0$



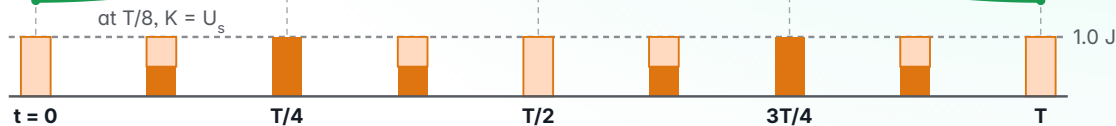
7.3 Position x
 ± 0.10 m

7.3 Velocity v
 ± 2.0 m/s

7.3 Acceleration a
 ± 40 m/s², $F_{\text{net}} \pm 20$ N

7.4 Energy

$\square$ K $\square$ U_s
total always 1.0 J



The net force is proportional to the displacement and points the other way, so the acceleration graph is the position graph upside down. Pulling the cart out farther would not change the 0.314 s period.

What the exam asks most

- Energy bars, graphs and a force diagram that agree.** A question gives you one of these, such as a position graph, and asks you to draw another at a marked time and explain why they agree. At either end the cart stops for an instant, all the energy is in the spring, and the net force is largest, pointing back toward the middle. At $x = 0$ the cart is fastest, all the energy is kinetic, and the net force is zero. (7.3) (7.4)

- How the period changes, as a ratio.** Write the ratio:
 $T_2/T_1 = \sqrt{m_2/m_1} = \sqrt{4} = 2$, so 4 times the mass doubles the period and halves the frequency. A bigger amplitude gives more energy and a higher top speed, but the same period. (7.2)

- A spring constant found with a stopwatch.** Time 10 full cycles and divide by 10 to get T . Repeat with several masses and graph T^2 against m . Since $T^2 = (4\pi^2/k)m$, the points fall on a straight line and $k = 4\pi^2/\text{slope}$. (7.2)

- The spring constant and mass, read off graphs.** If the graphs show a spring force of 6.0 N when the cart is 0.12 m from the middle, $k = 6.0/0.12 = 50$ N/m. A period of 0.80 s then gives $m = kT^2/4\pi^2 = 0.81$ kg. (7.1) (7.2)

- A speed from energy.** The total energy is shared:
 $\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. For the cart above at $x = 0.050$ m, the spring holds $\frac{1}{2}(200)(0.050)^2 = 0.25$ J of the 1.0 J, so $\frac{1}{2}(0.50)v^2 = 0.75$ J and $v = 1.7$ m/s. (7.4)

On the equation sheet

$$T = \frac{1}{f} \quad T_s = 2\pi\sqrt{\frac{m}{k}}$$

$$x = A \cos(2\pi ft) \quad T_p = 2\pi\sqrt{\frac{\ell}{g}}$$

$$x = A \sin(2\pi ft) \quad U_s = \frac{1}{2}k(\Delta x)^2$$

$$\vec{F}_s = -k\Delta x \quad K = \frac{1}{2}mv^2$$

Use $g = 10$ m/s²: exam questions that need a number for g use 10, and so does every example here. The equation sheet you get on exam day lists 9.8 m/s², and answers worked with 9.8 or 9.81 also get full credit.

Not on the equation sheet: know it by heart

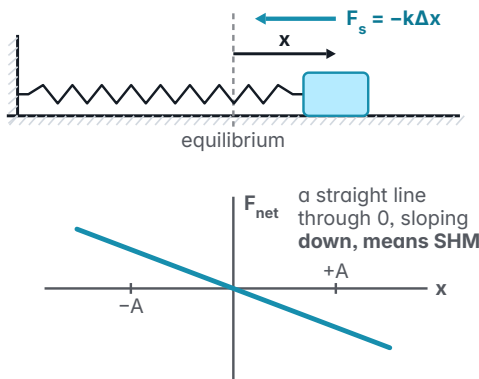
- $F_{\text{net}} = -kx$, with x measured from equilibrium. On a level track this is the spring force; on a hanging spring it is the spring force plus the weight.
- On a level track, $E = \frac{1}{2}kA^2 = K + U_s$ (kinetic plus spring energy). At $x = 0$, $U_s = 0$, so $\frac{1}{2}kA^2 = \frac{1}{2}mv_{\text{max}}^2$ and $v_{\text{max}} = A\sqrt{k/m}$.
- $f = 1/T_s = \frac{1}{2\pi}\sqrt{k/m}$.

Not required in this course

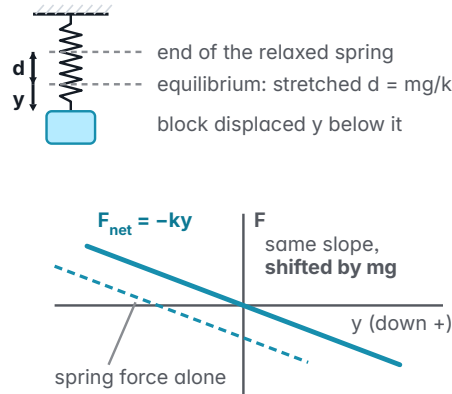
You won't be asked to calculate the period of a pendulum at a large angle, of a swinging rod or a twisting wire, or how fast friction makes the swings die down.

7.1 Defining simple harmonic motion

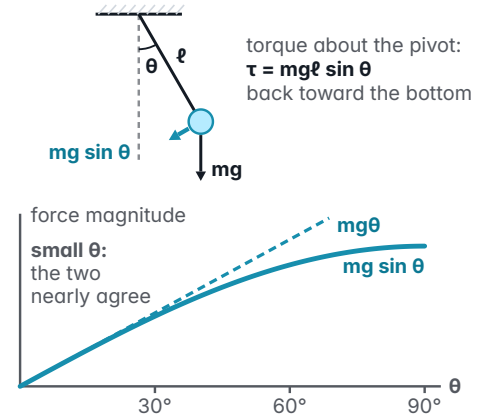
Cart on a spring



Hanging spring



Pendulum, small angle



A net-force graph that is a straight line through zero with a negative slope means SHM. Hanging the spring shifts the spring-force line (dashed) by mg , but the net-force line still passes through zero. A pendulum's $mg \sin \theta$ stays close to $mg\theta$ (θ in radians) only while the angle is small.

Anything that repeats is **periodic**. It is **simple harmonic motion (SHM)** only when the force pulling it back is proportional to the displacement.

What the words mean

- Equilibrium:** the middle, where the net force is zero.
- Displacement x :** how far the object is from equilibrium, + on one side and - on the other.
- Restoring force:** the net force along the path, which always points back toward equilibrium.
- Amplitude A :** the largest displacement, measured from the middle, not from end to end.

Where $ma = -kx$ comes from

The equation sheet you get on exam day gives Newton's second law, $\vec{a}_{\text{sys}} = \sum \vec{F}/m_{\text{sys}}$, and Hooke's law, $\vec{F}_s = -k\Delta\vec{x}$. With the spring force as the net force, $ma = -kx$, so $a = -(k/m)x$: acceleration proportional to displacement and opposite to it, which is SHM. That step isn't on the sheet, so write it out.

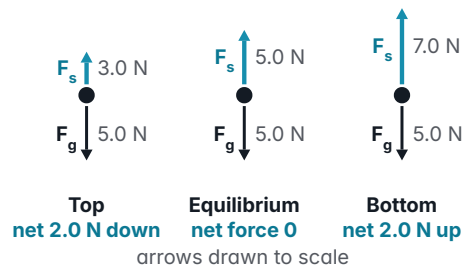
Which spring equation to use

The question asks for	Start from
the spring's force at one moment	$\vec{F}_s = -k\Delta\vec{x}$
an energy, or a speed	$U_s = \frac{1}{2}k(\Delta x)^2$
the period or frequency	$T_s = 2\pi\sqrt{m/k}$

A isn't in T_s , so a bigger swing doesn't take longer. Δx is the stretch or squeeze from the relaxed length; on a hanging spring that isn't y , so count gravity's energy too.

The hanging spring

Gravity stretches a hanging spring by d , where $kd = mg$, and the block rests there. If y is the displacement from that resting point (down positive), the spring force and the weight add up to $-k(d + y) + mg = -ky$, since $kd = mg$. So it is the same SHM, with the same period T_s , as on a level track.



The spring stays stretched at all three points, so it always pulls up, with more force than the weight below equilibrium and less above it.

Worked example

A 0.50 kg block hangs on a 50 N/m spring. It is pulled 0.040 m below its resting point and let go. Find the resting stretch, the forces at the lowest point and the acceleration there.

- At rest, $kd = mg$, so $d = (0.50)(10)/50 = 0.10 \text{ m}$.
- At the lowest point the stretch is $0.10 + 0.040 = 0.14 \text{ m}$, so $F_s = (50)(0.14) = 7.0 \text{ N}$ up. The weight is 5.0 N down.
- $F_{\text{net}} = 7.0 - 5.0 = 2.0 \text{ N}$ up, which matches $kA = (50)(0.040)$.

$a = 2.0 \text{ N} / 0.50 \text{ kg} = 4.0 \text{ m/s}^2$ up, toward equilibrium. The period is $T_s = 2\pi\sqrt{0.50/50} = 0.63 \text{ s}$, as on a level track.

Drawing the free-body diagram

Draw one arrow for each force, starting on the dot. Don't add a "restoring force" arrow: on a hanging spring the restoring force is the spring force and the weight added together. At the lowest point the spring arrow is longer than the weight arrow.

Exam trap

- Thinking the block is in equilibrium at the lowest point, because it stops there.** Equilibrium means the net force is zero, and stopping for an instant doesn't make it zero. In the worked example the block is at rest at the lowest point, yet the net force there is 2.0 N up, the largest it ever gets.
- Thinking a falling block slows down as soon as it lands on a spring.** At first the spring pushes up less than the weight, so the net force is still down and the block keeps speeding up. It is fastest where the spring force equals the weight, $k\Delta x = mg$ (0.10 m into a 50 N/m spring for a 0.50 kg block), and slows only after that.

Quick check

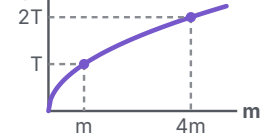
Which of these are SHM? (a) a cart on a horizontal spring; (b) a ball bouncing on the floor; (c) a pendulum swinging 5° each side; (d) the same pendulum swinging 70° each side.

(a) Yes: the spring force grows with x and points back. (b) No: in the air the only force is its constant weight, which doesn't grow with distance. (c) Yes: at 5° , $\sin \theta$ and θ differ by about 0.1%. (d) No: at 70° , $\sin \theta$ is far from θ , and T_p doesn't apply.

7.2 Frequency and period of SHM

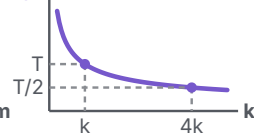
Spring: $T_s = 2\pi\sqrt{(m/k)}$

T_s against m



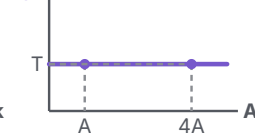
$m \times 4$ gives $T \times 2$

T_s against k



$k \times 4$ gives $T \times \frac{1}{2}$

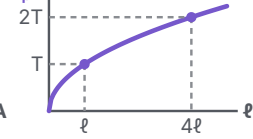
T_s against A



no effect from A

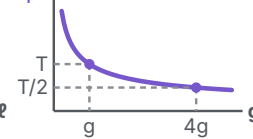
Pendulum: $T_p = 2\pi\sqrt{(\ell/g)}$

T_p against ℓ



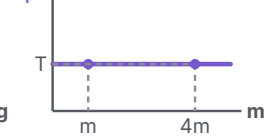
$\ell \times 4$ gives $T \times 2$

T_p against g



$g \times 4$ gives $T \times \frac{1}{2}$

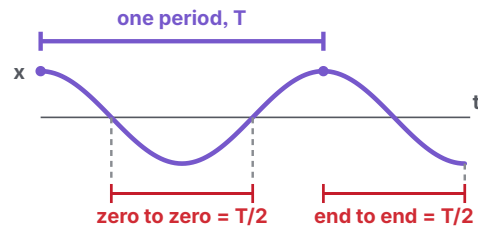
T_p against bob mass



no effect from mass

Every quantity that changes the period sits under a square root. Multiplying one by 4 doubles the period if it is on top (m , ℓ) and halves it if it is on the bottom (k , g). The amplitude and a pendulum bob's mass aren't in the formulas at all.

The **period** T is the time for one full cycle, in seconds. The **frequency** f is the number of cycles per second, in hertz (Hz). Each is 1 over the other: a swing with $T = 0.50$ s has $f = 2.0$ Hz.



A cycle ends only when the object is back at the same place, moving the same way, so read T from one peak to the next. Zero to zero, or end to end, is only half a period.

On the equation sheet

$$T = \frac{1}{f} \quad T_s = 2\pi\sqrt{\frac{m}{k}} \quad T_p = 2\pi\sqrt{\frac{\ell}{g}}$$

The equation sheet doesn't give the spring's frequency. It is $f = 1/T_s = \frac{1}{2\pi}\sqrt{k/m}$. ℓ is measured from the pivot to the center of the bob, and T_p works only for small angles.

Exam trap

- **Timing from one end to the other and calling it the period.** That is only half a cycle, so double it: a block that takes 0.40 s from the top of its bounce to the bottom has a period of 0.80 s.
- **Writing $f = 2\pi\sqrt{(m/k)}$.** That formula gives the period. The frequency is its reciprocal: for 0.50 kg on a 200 N/m spring, $T = 0.31$ s, so $f = 1/T = 3.2$ Hz.
- **Using only the added mass.** The spring moves the cart and the block together. A $3m_0$ block on an m_0 cart makes $4m_0$, which doubles the period.
- **Saying a bigger g gives a longer period.** g is on the bottom of the fraction in T_p , so a bigger g gives a shorter period: four times g halves a pendulum's period.

How the period changes

If you multiply	Spring's T	Pendulum's T
mass by 4	$\times 2$	same
mass by 2	$\times 1.41$ ($\sqrt{2}$)	same
k by 4	$\times \frac{1}{2}$	not in T_p
ℓ by 4	not in T_s	$\times 2$
g by 4	same	$\times \frac{1}{2}$
amplitude by 2	same	same

The frequency changes by the opposite factor: when the period doubles, the frequency halves. g only moves the point where a hanging spring rests, so it doesn't change T_s .

Worked example

A cart of mass m_0 on a spring oscillates with frequency f_1 . A block of mass $8m_0$ is stuck onto it. Find f_2/f_1 .

- 1 The same spring now moves $9m_0$ in all. $T_s \propto \sqrt{m}$, so $\frac{T_2}{T_1} = \sqrt{\frac{9m_0}{m_0}} = 3$.
- 2 $f = 1/T$, so the frequency ratio is the reciprocal.

$$f_2/f_1 = 1/3$$

Worked example

The spring force on a block reaches 6.0 N when the block is 0.12 m from equilibrium. Its position graph repeats every 0.80 s. Find k and the block's mass.

- 1 Read the force and the displacement at the same moment: $k = 6.0/0.12 = 50$ N/m.
- 2 Square the sheet's $T_s = 2\pi\sqrt{m/k}$ and solve: $m = kT^2/4\pi^2$.
- 3 $m = \frac{(50)(0.80)^2}{4\pi^2} = \frac{32}{39.5}$

$$k = 50 \text{ N/m}, m = 0.81 \text{ kg}$$

When the pendulum formula works

θ	θ in radians	$\sin \theta$	true period is longer than T_p by
10°	0.1745	0.1736	0.2%
20°	0.3491	0.3420	0.8%
30°	0.5236	0.5000	1.7%
60°	1.0472	0.8660	7%

Gravity's torque about the pivot is $mg\ell \sin \theta$. SHM needs it proportional to θ , which is close to true only while $\sin \theta \approx \theta$ (θ in radians). The course sets no cutoff angle; a question will just say the angle is small.

A stopwatch can measure length and mass

A rope swing that takes 4.0 s per cycle has length $\ell = gT^2/4\pi^2 = (10)(4.0)^2/4\pi^2 = 4.1$ m. In orbit a bathroom scale reads zero, but a spring's period still depends on mass, $m = kT^2/4\pi^2$. Skylab astronauts measured their body mass this way, on a chair mounted on springs.

Worked example

An oscillating cart passes $x = 0$ moving toward $+x$ at $t = 0.20$ s, toward $-x$ at $t = 0.575$ s, and toward $+x$ again at $t = 0.95$ s. Find T and f .

- 1 Go from one pass toward $+x$ to the next: $T = 0.95 - 0.20 = 0.75$ s.
- 2 The pass toward $-x$ at 0.575 s is only half a period in.

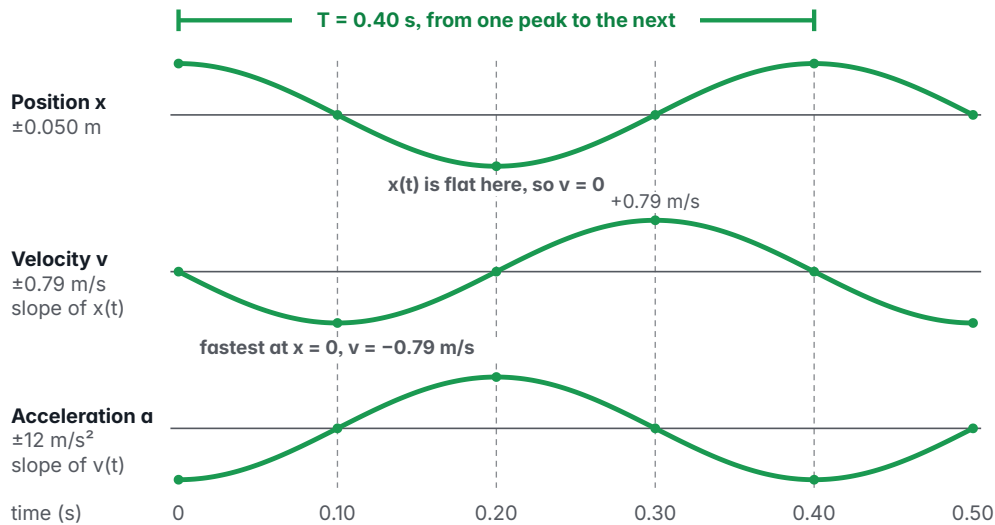
$$T = 0.75 \text{ s and } f = 1/0.75 = 1.3 \text{ Hz}$$

Quick check

A pendulum and a hanging spring each have $T = 1.0$ s on Earth. What are their periods on a planet with Earth's radius and 4 times its mass?

$g = GM/R^2$, so with the same R and 4 times the mass, g is 4 times as big: T_p halves to 0.50 s. T_s stays 1.0 s, because g isn't in it.

7.3 Representing and analyzing SHM



The cart moves as $x = (0.050 \text{ m}) \cos(2\pi \cdot 2.5 t)$, so $A = 0.050 \text{ m}$, $f = 2.5 \text{ Hz}$ and $T = 0.40 \text{ s}$. The acceleration peaks at the ends at $(k/m)A = (2\pi/T)^2 A = 12 \text{ m/s}^2$.

To find	Look at
amplitude A	the height of a peak of $x(t)$ above the axis, not peak to trough
period T	one peak of $x(t)$ to the next
velocity	the slope of $x(t)$
acceleration	the slope of $v(t)$, or $a = -kx/m$
displacement	the area under $v(t)$

Worked example

For the cart above, find x at $t = 0.050 \text{ s}$ and the speed there.

- In radian mode, $x = 0.050 \cos(2\pi \cdot 2.5 \cdot 0.050) = 0.050 \cos(\pi/4) = 0.035 \text{ m}$.
- From $T_s = 2\pi\sqrt{m/k}$, $\sqrt{k/m} = 2\pi/T = 15.7 \text{ s}^{-1}$.
- Energy is conserved, $\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, so $v = \sqrt{k/m} \sqrt{A^2 - x^2} = 15.7\sqrt{0.050^2 - 0.035^2}$.

$v = 0.56 \text{ m/s}$, heading toward $-x$. At $x = 0$ the same equation gives the top speed, $(15.7)(0.050) = 0.79 \text{ m/s}$.

Exam trap

- Calling the distance from one end to the other the amplitude. The amplitude is measured from the middle. The cart above swings 0.10 m from end to end, so $A = 0.050 \text{ m}$, and in one full cycle it travels $4A = 0.20 \text{ m}$.
- Working out ΔK as $\frac{1}{2}m(\Delta v)^2$. K depends on speed squared, so square each speed first: $\Delta K = \frac{1}{2}m(v_f^2 - v_i^2)$. In the example below, squaring $\Delta v = -0.80 \text{ m/s}$ gives 0.128 J , 4 times too big and with the wrong sign; the right answer is -0.032 J .

Worked example

The velocity graph of a 0.40 kg cart on a spring goes from $+0.50 \text{ m/s}$ to -0.30 m/s between two times. Find ΔK and Δp .

- $\Delta K = \frac{1}{2}(0.40) [(0.30)^2 - (0.50)^2] = -0.032 \text{ J}$
- $\Delta p = (0.40)(-0.30 - 0.50) = -0.32 \text{ kg}\cdot\text{m/s}$

$\Delta K = -0.032 \text{ J}$ and $\Delta p = -0.32 \text{ kg}\cdot\text{m/s}$. Over a whole period both are zero.

The cart above, every quarter period

	$t = 0$	$T/4 = 0.10 \text{ s}$	$T/2 = 0.20 \text{ s}$	$3T/4 = 0.30 \text{ s}$
x	$+0.050 \text{ m}$	0	-0.050 m	0
v	0	-0.79 m/s	0	$+0.79 \text{ m/s}$
a	-12 m/s^2 (largest magnitude)	0	$+12 \text{ m/s}^2$ (largest magnitude)	0
Energy	all in the spring	all kinetic	all in the spring	all kinetic

The equation sheet gives both $x = A \cos(2\pi ft)$ and $x = A \sin(2\pi ft)$. Use cosine when the object starts at an end, and sine when it starts in the middle moving toward $+x$. Starting at $-A$, or in the middle moving toward $-x$, adds a minus sign.

Let go from rest at $x = +A$

$$x = A \cos(2\pi ft)$$

Let go from rest at $x = -A$

$$x = -A \cos(2\pi ft)$$

At $x = 0$, moving $+x$, at $t = 0$

$$x = A \sin(2\pi ft)$$

$2\pi ft$ is in radians: in degree mode, $\cos(\pi/4)$ gives 0.99991 instead of 0.707 . Switch back to degree mode afterward, with the same steps, for ramps and force components.

Set radian mode

TI-84 Plus CE. `mode`, arrow down to the row Radian Degree, highlight Radian, press `enter`, then `2nd mode` to quit. The top of the screen now says Radian.

TI-Nspire CX. `on` for the home screen, then 5 Settings, 2 Document Settings. Set Angle to Radian, move to Make Default and press `enter`.

Desmos. Desmos starts in radians. If the wrench (graph settings) shows Degrees, switch it to Radians.

Exam trap

Finding Δp as the spring force at one instant times the elapsed time. A spring's force changes all the time, so one reading times Δt gives the wrong impulse. Use $\Delta p = m\Delta v$ from the velocity graph, or the area under the force-time graph. For the example cart, that area must equal $m\Delta v = -0.32 \text{ N}\cdot\text{s}$.

Quick check: which top speed is higher?

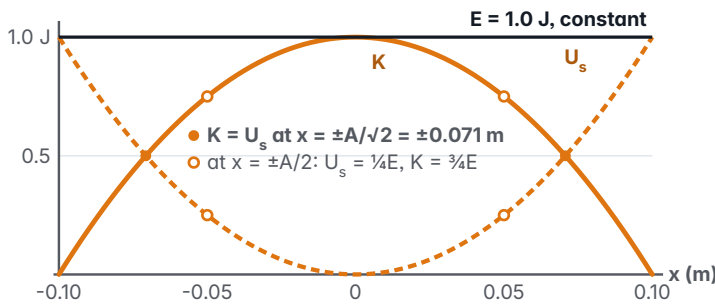
Cart 1 (solid): $A = 0.10 \text{ m}$, $T = 0.40 \text{ s}$
Cart 2 (dashed): $A = 0.050 \text{ m}$, $T = 0.20 \text{ s}$



Neither. $v_{\max} = A\sqrt{k/m} = 2\pi A/T$ gives 1.6 m/s for both: $2\pi(0.10)/0.40$ for cart 1 and $2\pi(0.050)/0.20$ for cart 2. Cart 1 swings twice as far but takes twice as long, so the two effects cancel.

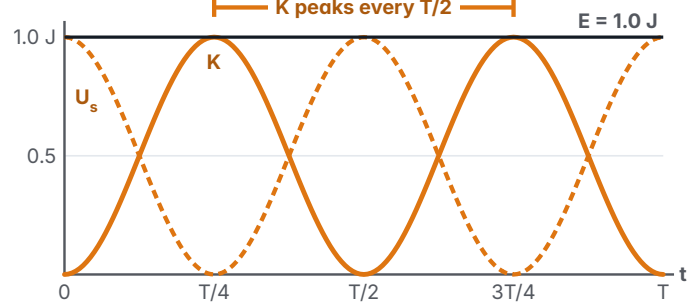
7.4 Energy of simple harmonic oscillators

Energy against position



The cart from page 1 (0.50 kg, 200 N/m, $A = 0.10$ m) has 1.0 J in all. U_s peaks at each end and K each time the cart passes $x = 0$, twice per cycle, so each energy repeats every $T/2$.

Energy against time



The energy equations

The equation sheet gives $U_s = \frac{1}{2}k(\Delta x)^2$ and $K = \frac{1}{2}mv^2$. With no friction their total stays the same, so:

- $E = K + U_s$, the same at every point of the motion.
- At an end the cart stops, so $E = \frac{1}{2}kA^2$.
- In the middle it is all kinetic: $\frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2$, so $v_{\max} = A\sqrt{k/m}$.

Worked example

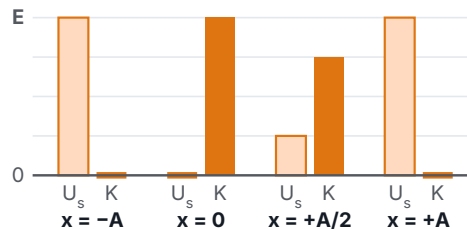
For the cart from page 1 (0.50 kg, 200 N/m, $A = 0.10$ m), find the speed and the acceleration at $x = 0.060$ m.

- $E = \frac{1}{2}(200)(0.10)^2 = 1.0$ J
- $U_s = \frac{1}{2}(200)(0.060)^2 = 0.36$ J, so $K = 0.64$ J.
- $v = \sqrt{2K/m} = \sqrt{2(0.64)/0.50}$ and $a = -kx/m = -(200)(0.060)/0.50$.

$v = 1.6$ m/s, in either direction; $a = -24$ m/s², toward $x = 0$.

Exam trap

- Saying K and U_s are equal all the time. Only their sum stays the same. They are equal only at $x = \pm A/\sqrt{2}$, which is ± 0.071 m for the cart above, where each is 0.50 J.
- Thinking twice the amplitude means twice the energy. $E = \frac{1}{2}kA^2$ depends on A squared, so doubling A gives $(2A)^2 = 4A^2$, four times the energy. Pulling the cart above out 0.20 m instead of 0.10 m gives 4.0 J, not 2.0 J.
- Saying a hanging spring stores no energy at equilibrium. It is stretched by mg/k there, so it stores $\frac{1}{2}k(mg/k)^2$, 0.25 J for a 0.50 kg block on a 50 N/m spring. Only the spring and gravity energies together, $\frac{1}{2}ky^2$ with y from equilibrium, are zero there.



Drawing energy bar charts

Draw only a K bar at $x = 0$ and only a U_s bar at each end, with the same total height at every position. Show an energy that is zero as a short line on the zero line, and keep the heights to scale.

Changing A , m or k

	E	$v_{\max}$	$a_{\max}$	T
$A \times 2$	$\times 4$	$\times 2$	$\times 2$	same
$m \times 4$	same	$\times \frac{1}{2}$	$\times \frac{1}{4}$	$\times 2$
$k \times 4$	$\times 4$	$\times 2$	$\times 4$	$\times \frac{1}{2}$

In each row, the other two of A , m and k stay the same. A heavier cart pulled out the same distance has the same energy but a lower top speed ($v_{\max} = A\sqrt{k/m}$, $a_{\max} = kA/m$).

Worked example

A pendulum bob is let go 5.0 cm above its lowest point. How fast does it pass the bottom?

- $mg\Delta y = \frac{1}{2}mv^2$, so the mass cancels.

$$v = \sqrt{2(10)(0.050)} = 1.0 \text{ m/s}$$

Quick check

Where is a spring oscillator moving at half its top speed?

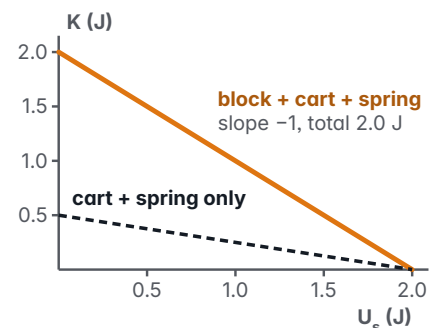
Half the speed means a quarter of the kinetic energy, $K = \frac{1}{4}E$. The spring holds the other $\frac{3}{4}E$, so $(x/A)^2 = \frac{3}{4}$ and $x = \pm 0.87A$, not $\pm A/2$.

Dropping a block on the cart

A 1.0 kg cart on a 100 N/m spring oscillates with $A = 0.20$ m, so $E = 2.0$ J, $T = 0.63$ s and $v_{\max} = 2.0$ m/s. A 3.0 kg block falls straight down onto it and sticks.

	After	Lands at an end	Lands at $x = 0$
E	2.0 J, same	0.50 J	
A	0.20 m, same	0.10 m	
T	1.26 s	1.26 s	
$v_{\max}$	1.0 m/s	0.50 m/s	

At an end the cart is at rest for an instant, so only the mass changes. Landing in the middle, the block must be brought up to the cart's speed: sideways momentum is kept and kinetic energy is lost. $(1.0)(2.0) = (4.0)v$ gives $v = 0.50$ m/s, so $K = \frac{1}{2}(4.0)(0.50)^2 = 0.50$ J and $A = \sqrt{2(E/K)} = 0.10$ m.



This is after the block lands at an end. For block, cart and spring together, $K + U_s = 2.0$ J at every moment, so the line meets both axes at 2.0 J. For the cart and spring alone, K is just the cart's $\frac{1}{4}$ share, so that line starts at 0.50 J.

Writing why the energy stayed the same

The block landed when the cart was at rest at an end. The cart had no kinetic energy to share, and the spring's stretch didn't change, so the oscillation energy $\frac{1}{2}kA^2$ stayed the same.

The block's own downward kinetic energy is lost, but it was never part of the oscillation. Writing only "energy is conserved" doesn't earn the point.

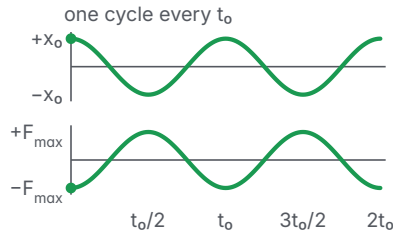
Free-response questions 1 and 2

Question 1: Mathematical Routines 10 points

You derive an equation, usually from one on the equation sheet, and calculate with units. In this unit that might be a period, a frequency or a speed.

⊗ A question like question 2, worked point by point

A cart on a horizontal spring is let go from rest at $x = +x_0$ at $t = 0$. The graphs show its position and the spring force on it; one cycle takes t_0 . (A) Draw energy bars at $+x_0$, 0 and $-x_0$. (B) Derive the cart's mass in terms of $F_{\max}$, x_0 and t_0 . (C) Sketch $v(t)$. (D) A student's free-body diagram for $t = 3t_0/4$ shows a net force to the right. Is it consistent with your $v(t)$? Each row below is one point, as in the course's sample spring question.



Part	An answer that earns the point	Why it earns the point
A	At $x = 0$, a single K bar and no U_s bar.	In the middle all the energy is kinetic.
A	At $\pm x_0$, a single U_s bar and no K bar.	At the ends the cart is at rest for an instant.
A	The same total height at all three places.	No friction, so the total energy stays the same.
B	$k = F_{\max}/x_0$	Read the force and the displacement at the same instant, at an end.
B	$T_s = 2\pi\sqrt{m/k}$, written first.	It is the equation from the sheet. A formula for m with no starting point loses this point.
B	$t_0 = 2\pi\sqrt{mx_0/F_{\max}}$	Substituting either your k or $T = t_0$ earns this point.
B	$m = \frac{F_{\max} t_0^2}{4\pi^2 x_0}$	Square both sides and solve, using only $F_{\max}$, x_0 and t_0 .
C	A curve that repeats every t_0 , like the given graphs.	The velocity repeats exactly when the position does.
C	Peaks of equal size above and below zero.	The cart is as fast going one way as the other.
C	Starts at 0 and goes negative first.	Let go at $+x_0$, the cart moves toward $-x$ first; a curve that starts upward loses it.
D	"Not consistent." At $3t_0/4$ my $v(t)$ is at its peak, so its slope is zero.	Say whether it is consistent, and name the feature of your own graph you used.
D	"Zero slope means $a = 0$, so the net force is zero, not to the right."	It links the slope to the force through $a = F_{\text{net}}/m$. Part D is scored on its own, so right reasoning about a wrong sketch still earns both points.

⊗ Other prompts, and what to write

"Calculate f_2/f_1 " after a block is added	Divide the period equations using the total mass: $T_2/T_1 = \sqrt{4m_0/m_0} = 2$, so $f_2/f_1 = 1/2$.	7.2
"Why are the intercepts equal?" on a graph of K against U_s	For the whole system the total stays the same: all kinetic in the middle and all in the spring at the ends, so $K_{\max} = U_{s,\max} = E$.	7.4
"Why is the energy the same after the block lands?"	It landed at an end, where the cart was at rest, so the landing took no energy from the oscillation and the stretch didn't change: $\frac{1}{2}kA^2$ is the same.	7.4
"Find ΔK and Δp from the graphs"	$\Delta K = \frac{1}{2}m(v_f^2 - v_i^2)$, squaring each speed first. Δp is the area under the force-time graph. Over one whole period both are zero.	7.3

Question 2: Translation Between Representations 12 points

You are shown one oscillator in one form and draw it in others (energy bars, a graph, a force diagram) so they agree, then explain why.

⚡ How to write a derivation

Write the sheet's equation first, $T_s = 2\pi\sqrt{m/k}$. Then replace each symbol with what the question gives, such as $k = F_{\max}/x_0$ and $T = t_0$, and solve. Each step can earn a point; a final answer alone is not a derivation.

⚡ Which equation to start from

If it asks for	Start from
a time or period	T_s , T_p or $T = 1/f$
a speed	energy, $\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$
a force or acceleration	$\vec{F}_s = -k\Delta x$, then $a = F_{\text{net}}/m$
a block that sticks	momentum during the collision, then energy afterward
one thing changing	a ratio, with what stays fixed: $T_2/T_1 = \sqrt{m_2/m_1}$ for the same k

⚠ Exam trap

- **Giving a ratio upside down.** If a question asks for f_2/f_1 after the mass goes up 4 times, the answer is $1/2$, not 2. Write the ratio in the order asked, then check it: more mass means a lower frequency, so f_2/f_1 must be less than 1.
- **Leaving symbols the question didn't give.** A mass derived "in terms of $F_{\max}$, x_0 and t_0 " can't contain k or T . Substitute $k = F_{\max}/x_0$ and $T = t_0$ before you stop.

📖 What the command words ask for

- **Calculate:** show the steps, the numbers you put in and the unit of the answer.
- **Derive:** start from a law or an equation on the sheet and show each step, in symbols.
- **Determine:** reach an answer by reasoning or by calculating.
- **Justify:** give the physics that supports your claim, such as a law, an equation or a feature of a graph.
- **Sketch:** get the period, the peaks and the start right; exact values aren't needed.

Free-response questions 3 and 4

Question 3: Experimental Design and Analysis 10 points

You plan a way to find a quantity such as k or g with the equipment listed, choose what to graph so the points fall on a straight line, and use the slope to get the answer.

Squaring $T_s = 2\pi\sqrt{m/k}$ gives $T^2 = (4\pi^2/k)m$, so a graph of T^2 against m is a straight line through the origin with slope $4\pi^2/k$. A stopwatch and a scale are enough to find k .

Finding k with only a stopwatch and a scale

- 1 Measure each mass m on the scale.
- 2 Hang it from the spring, pull it down a little and let go.
- 3 Time 10 full cycles and divide by 10 to get T . Start and stop as the mass passes its resting point: it moves fastest there, so the moment is sharpest.
- 4 Repeat with four or five masses, timing each one more than once.
- 5 Graph T^2 against m , draw a best-fit line, and use $k = 4\pi^2/\text{slope}$.

The rubric points come from using the scale for the mass and the stopwatch for the period, setting the mass oscillating, and a step that cuts uncertainty, such as timing many cycles.

Why time 10 cycles instead of one

Hand timing can be off by about 0.2 s. On one 0.89 s cycle that is more than 20%; over 10 cycles (8.9 s) it is about 2%.

Exam trap

Planning to measure the stretch when there is no ruler. With only a stopwatch and a scale you can't measure how far the spring stretches, so a plan built on $k = mg/\text{stretch}$ earns nothing. Time the oscillations instead.

Finding g with a pendulum

Do the same with string lengths: graph T^2 against ℓ and use $g = 4\pi^2/\text{slope}$. Measure ℓ from the pivot to the center of the bob, and keep the swings small.

What earns points on the graph

Label each axis with the quantity and its unit: T^2 (s^2) and m (kg). Draw one straight best-fit line with points on both sides of it, not a line joining the dots. Take the slope from two points on your line that are far apart, then use $k = 4\pi^2/\text{slope}$, with its unit, N/m.

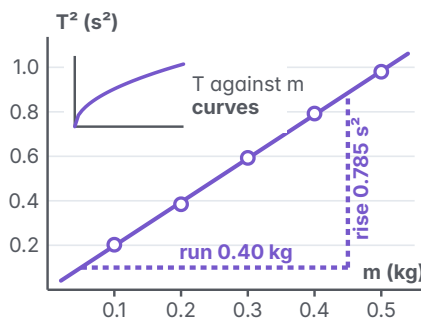
Worked example

Five masses hang in turn from one spring. Find k from these sample timings.

m (kg)	10 cycles (s)	T (s)	T^2 (s^2)
0.10	4.5	0.45	0.20
0.20	6.2	0.62	0.38
0.30	7.7	0.77	0.59
0.40	8.9	0.89	0.79
0.50	9.9	0.99	0.98

- 1 Plot T^2 against m and draw one straight best-fit line.
- 2 Read two points on the line, far apart: (0.05 kg, 0.100 s^2) and (0.45 kg, 0.885 s^2). Slope = $0.785/0.40 = 1.96$ s^2/kg .

$$k = 4\pi^2/1.96 = 20 \text{ N/m}$$



The rise and run are read from the line at $m = 0.05$ kg and 0.45 kg, away from the data points. The small graph shows why T^2 is used: T itself grows as $\sqrt{m}$, so its graph against m curves.

Graph	Its slope is	Then
T^2 against m	$4\pi^2/k$	$k = 4\pi^2/\text{slope}$
m against T^2	$k/4\pi^2$	$k = 4\pi^2 \times \text{slope}$
T^2 against ℓ	$4\pi^2/g$	$g = 4\pi^2/\text{slope}$

Quick check

A graph of T^2 against ℓ for a pendulum has a slope of 4.0 s^2/m . What is g ?

$$T_p^2 = (4\pi^2/g)\ell, \text{ so the slope is } 4\pi^2/g \text{ and } g = 4\pi^2/4.0 = 9.9 \text{ m/s}^2.$$

Check the slope on your calculator

TI-84 Plus CE. Press **[stat]** and choose 1: Edit. Enter the masses in L1 and the T^2 values in L2. Then **[stat]** $\rightarrow$ CALC $\rightarrow$ 4: LinReg(ax+b), set Xlist L1 and Ylist L2, and choose Calculate. The slope is a .

TI-Nspire CX. On a Lists & Spreadsheet page, type x and y as the column names in the top row, then enter the masses under x and the T^2 values under y . **[menu]** $\rightarrow$ Statistics $\rightarrow$ Stat Calculations $\rightarrow$ Linear Regression ($mx+b$), with X List x and Y List y . The slope is m .

Desmos. Add a table (+ $\rightarrow$ table) with the masses in x_1 and the T^2 values in y_1 , then type $y_1 \sim mx_1 + b$. Desmos prints m and b .

For the table here the calculator gives 1.97 s^2/kg , close to the 1.96 read from the line. On the exam the line must still be drawn on your graph, and the slope worked out from it.

Question 4: Qualitative/Quantitative Translation 8 points

You explain in words what happens when one thing changes, then show the same result with equations, and say how the two agree.

Worked example

A cart oscillates on a spring. It is moved to a spring with 4 times the spring constant. How does its frequency change?

- 1 Say it in words first: the stiffer spring pulls harder at every displacement, so the cart's acceleration is larger and each cycle ends sooner. The frequency goes up.
- 2 Then show it with the equation: $f = \frac{1}{2\pi}\sqrt{k/m}$, so $f_2/f_1 = \sqrt{4} = 2$.

Both say the frequency goes up, and the equation shows it doubles.

A pendulum on an elastic string, on a heavier planet

The planet has the same radius and more mass, so g is larger. A larger g shortens the period, because g is on the bottom of $T_p = 2\pi\sqrt{\ell/g}$. But the bob's weight is larger too, so the string stretches, and a longer ℓ lengthens the period. A full answer names both effects and their directions:

The larger g shortens the period, but the stretched string lengthens it, so the period increases only if ℓ grows by a larger factor than g .