

AP Precalculus

Whole-course cheat sheet

The exam is on
Tuesday, May 11, 2027

How the exam works in May 2027

Part	Questions	Time	Share of your score	Calculator
Multiple choice, Part A	29	65 minutes	43.75%	Not allowed
Multiple choice, Part B	13	40 minutes	18.75%	Graphing calculator required
Free response, Part A (questions 1 and 2)	2	35 minutes	18.75%	Graphing calculator required
Free response, Part B (questions 3 and 4)	2	35 minutes	18.75%	Not allowed
The whole exam	46	2 hours 55 minutes	100%	

You answer multiple choice in the Bluebook app. The free-response questions appear there too, but you handwrite the answers in a paper booklet. A wrong multiple-choice answer costs no points, so never leave one blank. The two no-calculator parts make up 62.5% of your score.

The four units

Unit 1

Polynomial and Rational Functions

30–40% of the multiple-choice section, and free-response questions 1 and 2

Unit 2

Exponential and Logarithmic Functions

25–40% of the multiple-choice section, and free-response questions 1, 2 and 4

Unit 3

Trigonometric and Polar Functions

30–35% of the multiple-choice section, and free-response questions 3 and 4

Unit 4

Functions Involving Parameters, Vectors, and Matrices

Not on the AP exam. Your teacher may still test it in class.

The four free-response questions, 6 points each

Question 1: Function concepts, with a calculator

Functions are given as a graph, a table and a formula. You combine them (composition), find inverse values and zeros, describe end behavior or an asymptote with a limit, and choose a function type with a reason. It draws on Units 1 and 2.

Question 3: Modeling a periodic context, no calculator

Something that repeats, like a turning wheel or a tide. You give the coordinates of five points on its graph, find the constants of a sine or cosine model, and describe how the function behaves on an interval. It draws on Unit 3.

Question 2: Modeling a non-periodic context, with a calculator

For a real situation, you find a model's constants from data (polynomial, piecewise, exponential or logarithmic), explain an average rate of change with units, and explain where the model stops making sense. It draws on Units 1 and 2.

Question 4: Symbolic manipulations, no calculator

Exponential, logarithmic and trig functions. You solve equations exactly, showing every step, and rewrite expressions in a requested equivalent form. It draws on Units 2 and 3.

What you get on exam day

No formula sheet. The exam supplies no formula reference, so you need to know the formulas for Units 1–3. Unit 4 is not tested. A question may give a model's form, such as $V(t) = ab^t$, and ask you to find a and b .

A calculator in the calculator parts only. Bluebook has a built-in Desmos graphing calculator, and you may also bring up to two approved handheld graphing calculators of your own (a TI-84 Plus CE or a TI-Nspire CX, for example). The Desmos website and app are not allowed.

Decimals to three places. Unless a question says otherwise, decimals must be correct to three places after the decimal point. Store values in the calculator and round only at the end.

Exact answers without a calculator. In Part B of the free response, give answers like $\frac{\pi}{6}$, $\ln 4$ or $\frac{\sqrt{3}}{2}$, simplified as far as they go: $\log_2 8$ should become 3.

Set radian mode before the calculator parts

TI-84 Plus CE. `[mode]`, arrow down to the row Radian DEGREE, highlight Radian, press `[enter]`, then `[2nd] [mode]` to quit. The top of the screen now says Radian.

TI-Nspire CX. `[on]` for the home screen, then 5 Settings, 2 Document Settings. Set Angle to Radian, move to Make Default and press `[enter]`.

Desmos. Desmos starts in radians. If the wrench (graph settings) shows Degrees, switch it to Radians.

Store values so the third decimal stays right

TI-84 Plus CE. After a result, press `[sto→] [alpha] [math]` (the letter A) and `[enter]`. Use A later with `[alpha] [math]`.

TI-Nspire CX. After a result, press `[ctrl] [var] (sto→)`, type a name such as a, and press `[enter]`. Type the name to use it later.

Desmos. Define the value once, for example $a = (34 - 25)/4$, and use it everywhere else. Desmos keeps every digit.

Unit 1

Polynomial and Rational Functions

This unit is about how a function's output changes as its input changes. You measure change with average rates, read the zeros and end behavior of polynomials, find the asymptotes and holes of rational functions, move graphs with transformations, and build models from data.

On the exam: 30–40% of the multiple-choice section, plus free-response questions 1 and 2 (calculator allowed). Question 1 asks for zeros and end behavior; question 2 builds a model and asks where it stops making sense.

1.1 Change in tandem

A function gives exactly one output for each input. Describe it by what its outputs do as the input increases (they increase or decrease) and by whether that change speeds up or slows down. Concave up means the rate of change is increasing; concave down means it is decreasing.

Concave up: the rate of change is increasing



Concave down: the rate of change is decreasing



Worked example

A phone's charge is 10, 38, 60, 76 and 86 percent after 0, 10, 20, 30 and 40 minutes. Describe how it changes.

- Each 10-minute step adds 28, then 22, 16 and 10 percentage points.
- The gains shrink, so the rate of change is decreasing.

The data suggest the charge is increasing at a decreasing rate: a concave-down graph.

Common mistake

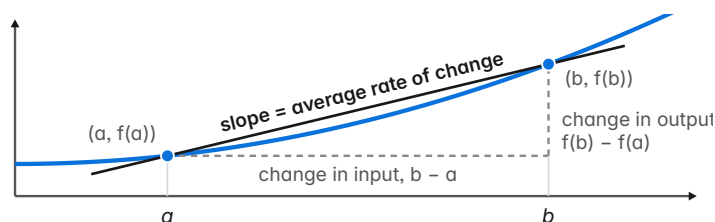
Calling a function increasing because its values are positive. Positive and negative say where the graph is, above or below the x -axis. Increasing and decreasing say which way it heads as x grows. On $(0, 2)$, $f(x) = x^2 - 4$ is negative and increasing: its values climb from -4 to 0 .

1.2 Rates of change

The average rate of change of f from $x = a$ to $x = b$ is the change in output divided by the change in input (Δ means "change in", so it is $\frac{\Delta y}{\Delta x}$). It is the slope of the secant line through the two points, in output units per input unit. Over smaller and smaller intervals around a point, it approaches the rate of change there, if that rate exists.

Formula

Average rate of change on $[a, b]$: $\frac{f(b) - f(a)}{b - a}$



Worked example

A rocket is 8.1 km up at $t = 60$ s and 20.0 km up at $t = 90$ s. Find the average rate of change and say what it means.

$$1 \quad \frac{20.0 - 8.1}{90 - 60} = \frac{11.9}{30} \approx 0.397 \text{ km per second}$$

From $t = 60$ to $t = 90$ seconds, the rocket's altitude increased by an average of 0.397 km per second.

Common mistake

Using "decreased" with a negative rate. Fuel that drops from 411 to 303 tonnes in 40 seconds changes by $\frac{303-411}{40} = -2.7$ tonnes per second. "Decreased at a rate of -2.7 " says down twice; write "decreased by 2.7 tonnes per second on average."

1.3 Rates of change in linear and quadratic functions

A linear function's average rate of change is the same on every interval. A quadratic's rate of change changes at a constant rate, so with equal input steps its second differences are constant (each is $2ah^2$ for $ax^2 + bx + c$ with steps of h).

Worked example

Stopping distances at 30, 40, 50 and 60 mph are 75, 120, 175 and 240 feet. Linear, quadratic or neither?

- The first differences are 45, 55 and 65. They are not constant, so the data aren't linear.
- The second differences are 10 and 10, which are constant.

Quadratic. (Check: $d = 0.05v^2 + v$ gives all four distances.)

Write it like this

The inputs go up in equal steps of 10 and the second differences are constant (10 and 10), so a quadratic model is appropriate.

1.4 Polynomial functions and rates of change

A nonconstant polynomial $a_n x^n + \dots + a_1 x + a_0$, with a_n not zero, has degree n and at most $n - 1$ turning points (local maxima and minima), and at least one between any two distinct real zeros. Where its rate of change switches between increasing and decreasing, it has a point of inflection.

Worked example

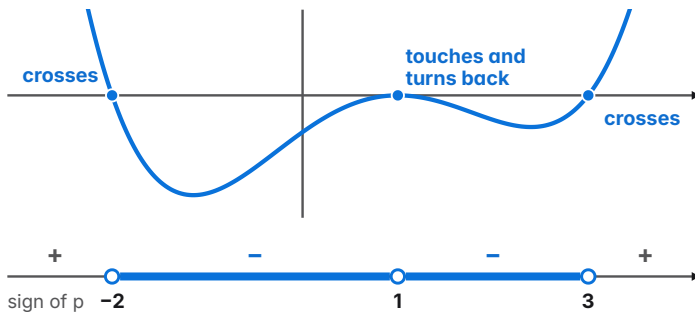
A cannonball is $h(t) = -16t^2 + 160t + 16$ feet high after t seconds. How high does it go?

- Vertex: $t = -\frac{b}{2a} = -\frac{160}{2(-16)} = 5$, and $h(5) = -400 + 800 + 16 = 416$.

It peaks at 416 feet, 5 seconds after it is fired.

1.5 Polynomial functions and complex zeros

If $p(a) = 0$, then $(x - a)$ is a factor of p , and for a real a the graph meets the x -axis at $(a, 0)$. A polynomial of degree n has exactly n complex zeros when a repeated zero is counted as many times as its factor appears (its multiplicity). If the coefficients are real, non-real zeros come in pairs: $a + bi$ and $a - bi$.



$p(x) = \frac{1}{2}(x+2)(x-1)^2(x-3)$. The squared factor $(x-1)^2$ is never negative, so p has the same sign on both sides of 1 and the graph only touches the axis there. At the single factors, $x = -2$ and $x = 3$, the sign flips and the graph crosses. So $p(x) < 0$ on $(-2, 1) \cup (1, 3)$, the thick band ($\cup$ means "or": either interval).

Formulas

Complex numbers $i^2 = -1$

Quadratic formula $ax^2 + bx + c = 0$ when $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Even, odd Even: $f(-x) = f(x)$, a mirror image in the y -axis. Odd: $f(-x) = -f(x)$, unchanged by a half-turn about the origin.

Worked example

Find a polynomial of least degree with real coefficients whose zeros include 3 (multiplicity 2) and $2 + i$.

1 Real coefficients mean $2 - i$ is a zero too.

2 $(x - 2 - i)(x - 2 + i) = (x - 2)^2 - i^2 = x^2 - 4x + 5$

$p(x) = (x - 3)^2(x^2 - 4x + 5)$, degree 4

Common mistake

Treating "even degree" as "even function". They are different ideas. $f(x) = x^2 + x$ has even degree, but $f(1) = 2$ and $f(-1) = 0$, so its graph is not symmetric about the y -axis. To test for even or odd, work out $f(-x)$ and compare it with $f(x)$.

1.6 Polynomial functions and end behavior

For inputs far from zero, a polynomial behaves like its leading term, the term with the highest power. If the degree is even, both ends of the graph point the same way; if it is odd, they point opposite ways. A positive leading coefficient a_n sends the right end up.



Write it like this

For $g(x) = 5 + 3x^2 - 2x^3$: as x increases without bound, $g(x)$ decreases without bound, so $\lim_{x \rightarrow \infty} g(x) = -\infty$.

A limit statement needs all four parts: lim, the approach ($x \rightarrow \infty$), the function and the value. Don't add an extra limit you haven't checked: a wrong one can cost the point.

Common mistake

Taking the first term written as the leading term. The leading term has the highest power, wherever it sits. $g(x) = 5 + 3x^2 - 2x^3$ leads with $-2x^3$, not 5: odd degree, negative coefficient, so the right end goes down. For a factored polynomial, multiply the leading parts: $(2x - 1)^3(4 - x)$ leads with $(2x)^3(-x) = -8x^4$, so both ends go down.

Find every zero

TI-84 Plus CE. Type the function in $y=$ and press **graph**. Press **2nd** **trace** (CALC), 2: zero, then **enter** left of a crossing, right of it, and once more. Repeat for each crossing. Where the graph only touches the axis this fails (NO SIGN CHNG): use 3: minimum or 4: maximum.

TI-Nspire CX. On a Graphs page, type the function. **menu** $\rightarrow$ Analyze Graph $\rightarrow$ Zero, then click to the left of a crossing and to the right of it. Where the graph only touches the axis, use Minimum or Maximum instead.

Desmos. Type the function and tap the curve: every zero, including one where the graph only touches the axis, shows as a grey dot. Tap a dot to read it.

1.7 Rational functions and end behavior

A rational function $r(x) = \frac{p(x)}{q(x)}$ behaves at its ends like the quotient of the leading terms of the top and the bottom, so compare their degrees.

End behavior of a rational function

Top has the lower degree asymptote $y = 0$

Same degree asymptote $y =$ ratio of the leading coefficients

Top one degree higher a slant line, from long division (1.11)

Top 2 or more higher no horizontal or slant asymptote; it acts like the quotient

Common mistake

Thinking a graph can't cross its horizontal asymptote. The asymptote only describes the ends. $r(x) = \frac{2x^2 + 8x}{x^2 + 4}$ has asymptote $y = 2$, yet solving $r(x) = 2$ gives $2x^2 + 8x = 2x^2 + 8$, so the graph crosses it at $x = 1$.

1.8 Rational functions and zeros

The real zeros of $r = \frac{p}{q}$ are the real zeros of the top that are not zeros of the bottom. To solve a rational inequality, get 0 on one side and one fraction on the other, mark where the top and the bottom are 0, and test one input in each interval. Zeros of the bottom are never included.

For example, $\frac{x^2 - 4}{x^2 + 3x + 2} = \frac{(x - 2)(x + 2)}{(x + 1)(x + 2)}$ has one zero, $x = 2$: at $x = -2$ the bottom is 0 too, so -2 is not in the domain.

Worked example

Solve $\frac{x + 4}{x - 1} \leq 3$.

- 1 Subtract 3: $\frac{x + 4 - 3(x - 1)}{x - 1} \leq 0$, so $\frac{7 - 2x}{x - 1} \leq 0$.
- 2 The boundary points are 3.5, which is included, and 1, which never is.
- 3 Test 0: $\frac{7}{-1} < 0$, yes. Test 2: $\frac{3}{1} > 0$, no. Test 4: $\frac{-1}{3} < 0$, yes.

$$(-\infty, 1) \cup [3.5, \infty)$$

Common mistake

Multiplying both sides by $x - 1$. If $x - 1$ is negative, the inequality sign has to flip, so this step quietly assumes $x > 1$. Here it gives only $x \geq 3.5$ and loses every $x < 1$. Use the one-fraction method above instead.

1.9 Rational functions and vertical asymptotes

First cancel any factor the top and the bottom share. If a factor $(x - a)$ is still in the bottom, the line $x = a$ is a vertical asymptote: near it, the output magnitudes grow without bound. Describe each side with a one-sided limit: $x \rightarrow a^-$ means inputs just below a , and $x \rightarrow a^+$ means inputs just above it.

Worked example

Find the one-sided limits of $r(x) = \frac{x - 4}{(x - 1)(x + 2)^2}$ at $x = 1$.

- 1 The other factors at $x = 1$: the top is $1 - 4 = -3$ and $(x + 2)^2 = 9$, so together they are negative.
- 2 Just right of 1, $x - 1$ is tiny and positive, so r is a negative number over a tiny positive one: hugely negative. Just left of 1, $x - 1$ is tiny and negative, so r is hugely positive.

$$\lim_{x \rightarrow 1^+} r(x) = -\infty \text{ and } \lim_{x \rightarrow 1^-} r(x) = \infty$$

Common mistake

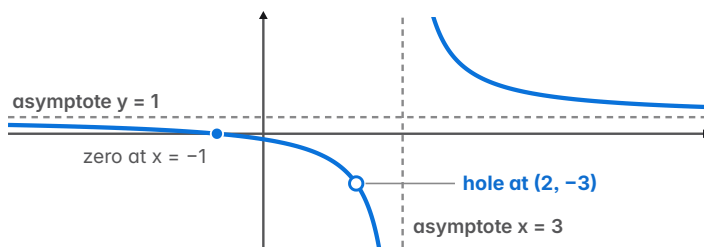
Assuming the sign flips at every asymptote. A squared factor like $(x - 2)^2$ is positive on both sides of 2, so it can't change the sign there. For $r(x) = \frac{x + 1}{(x - 2)^2}$, $r(1.9) = \frac{2.9}{0.01} = 290$ and $r(2.1) = \frac{3.1}{0.01} = 310$: the graph shoots up on both sides of $x = 2$. Only factors with an odd power flip the sign.

1.10 Rational functions and holes

If the factor $(x - c)$ cancels completely, because its power on top is at least its power on the bottom, the graph has a hole at $x = c$ instead of an asymptote. The hole's height is the value of the simplified expression at c , while $r(c)$ itself stays undefined.

Common mistake

Calling a hole on the x -axis a zero. $\frac{(x - 2)^2}{(x + 6)(x - 2)}$ simplifies to $\frac{x - 2}{x + 6}$, which is 0 at $x = 2$. But the original function is undefined at 2, so $(2, 0)$ is a hole, not a zero. Calculators draw straight through holes, so mark the open circle yourself.



$R(x) = \frac{(x - 2)(x + 1)}{(x - 2)(x - 3)}$. The factor $(x - 2)$ cancels, leaving $\frac{x + 1}{x - 3}$, which is -3 at $x = 2$: a hole at $(2, -3)$. The factor $(x - 3)$ stays on the bottom, so $x = 3$ is an asymptote. The ends approach $y = \frac{1}{-3} = -\frac{1}{3}$.

1.11 Equivalent representations of polynomial and rational expressions

One function can be written in several forms, and each answers different questions. Factored form shows zeros, holes and vertical asymptotes. Standard form shows the degree and the end behavior. Quotient plus remainder, from long division, shows a slant asymptote.

Formulas

Division

Dividing P by Q gives a quotient S and a remainder R : $P(x) = Q(x)S(x) + R(x)$, where R is 0 or has a lower degree than Q . So $\frac{P}{Q} = S + \frac{R}{Q}$.

Binomial theorem

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

add up these terms for $k = 0$ to n . Each coefficient $\binom{n}{k}$ is an entry in row n of Pascal's triangle, counting the top row as row 0; row 4 is 1, 4, 6, 4, 1.

Worked example

Find the slant asymptote of $\frac{2x^2 + 3x - 1}{x - 1}$.

- 1 $2x^2 \div x = 2x$, and subtracting $2x(x - 1)$ leaves $5x - 1$. Then $5x \div x = 5$, and subtracting $5(x - 1)$ leaves 4.
- 2 So it equals $2x + 5 + \frac{4}{x - 1}$, and the last part shrinks to 0 at both ends.

The slant asymptote is $y = 2x + 5$.

Worked example

Expand $(x - 2)^4$.

- 1 Row 4 is 1, 4, 6, 4, 1: $x^4 + 4x^3(-2) + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4$

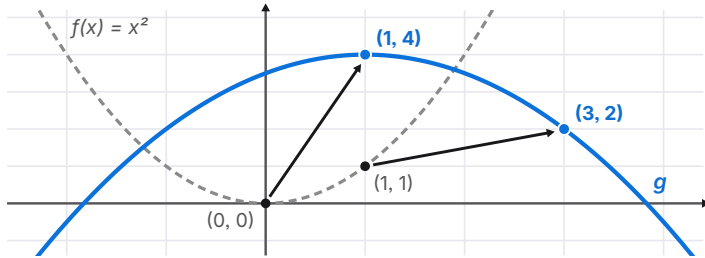
$x^4 - 8x^3 + 24x^2 - 32x + 16$. Check with $x = 1$: both sides give 1.

Common mistake

Raising only part of a term to the power. $(2x + 1)^3$ starts with $(2x)^3 = 8x^3$, not $2x^3$. And $(x + 2)^4$ is not $x^4 + 16$; it is $x^4 + 8x^3 + 24x^2 + 32x + 16$.

1.12 Transformations of functions

$g(x) = Af(B(x-h)) + k$, with $A, B \neq 0$, moves every point (u, v) on the graph of f to $(h + \frac{u}{B}, Av + k)$. Numbers outside f act on the outputs as written: multiply by A , then add k . Numbers inside act on the inputs in reverse: divide by B , then add h . Before the shifts, a negative A reflects across the x -axis and a negative B across the y -axis.



$f(x) = x^2$ (dashed) and $g(x) = -2f(\frac{1}{2}(x-1)) + 4$. The vertex $(0, 0)$ moves to $(1, 4)$, and $(1, 1)$ moves to $(1 + \frac{1}{1/2}, -2 \cdot 1 + 4) = (3, 2)$.

Common mistake

Reading the shift before factoring. $f(2x - 6)$ moves the graph right 3, not 6, because $2x - 6 = 2(x - 3)$. Factor out B first, then read h . And $f(x + 3)$ moves the graph left 3: plus inside means left.

1.13 Function model selection and assumption articulation

choosing a model, and what it assumes

Choose a model from how the data change. Roughly constant rates of change suggest a linear model. Rates that change at a roughly constant rate (constant second differences for equal input steps) suggest a quadratic. Several turns suggest a higher degree, and different behavior on different intervals calls for a piecewise model. Geometry helps too: for similar shapes, area is quadratic in length and volume cubic.

Every model rests on assumptions about what stays constant, and its domain comes from the context. Elapsed time can't be negative, and counts must be whole numbers: 1,128 students at 36 per bus need 32 buses, not 31.33.

Write it like this

Total plays of a song can never go down, but the song model worked out in Topic 1.14 decreases after its maximum at $t = 11$, so the model only makes sense for $0 \leq t \leq 11$.

Name the fact about the context and what the model does there. "Because of the vertex" on its own explains nothing.

1.14 Function model construction and application

Build a model from conditions (one equation for each data point), by moving a parent function into place, or with a regression on the calculator. Then use it: predict an output, solve for the input that gives a target value, and estimate with a secant line.

Common mistake

Starting the estimate line at the wrong point. The secant is the line through both endpoints of the interval. A line with the same slope through the middle data point, $(2, 30)$, is a different line: it gives $30 + 2.25(1.5 - 2) = 28.875$. Anchor the line at $(0, 25)$ or $(4, 34)$.

Formulas

Secant estimate $y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$, through $(a, f(a))$ and $(b, f(b))$

Over or under? Between a and b , a concave-down graph lies above its secant, so the estimate is too low. A concave-up graph lies below it, so the estimate is too high.

Inverse variation $y = \frac{k}{x}$, or $y = \frac{k}{x^2}$ for an inverse square (light, gravity, sound). One data point gives k .

Fit a quadratic to data

TI-84 Plus CE. **[stat]** 1: Edit. Enter the inputs in L1 and the outputs in L2. Then **[stat]** $\rightarrow$ CALC $\rightarrow$ 5: QuadReg, set Xlist L1 and Ylist L2, and choose Calculate.

TI-Nspire CX. On a Lists & Spreadsheet page, enter the data in columns named x and y . **[menu]** $\rightarrow$ Statistics $\rightarrow$ Stat Calculations $\rightarrow$ Quadratic Regression, with X List x and Y List y .

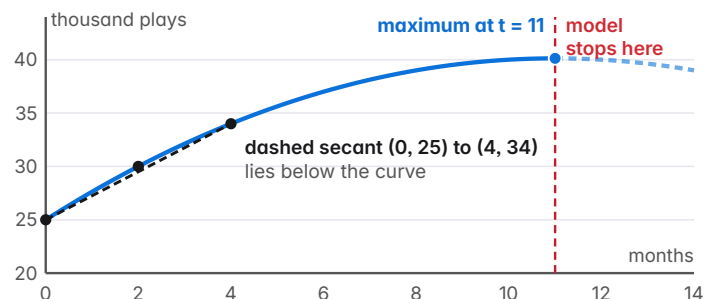
Desmos. Add a table (+ $\rightarrow$ table) with columns x_1 and y_1 , then type $y_1 \sim ax_1^2 + bx_1 + c$.

Worked example

A song's total plays, in thousands, are 25, 30 and 34 at $t = 0, 2$ and 4 months. Model them with $D(t) = at^2 + bt + c$, find and interpret the average rate of change over the first 4 months, and use it to estimate $D(1.5)$.

- One equation for each data point: $c = 25$, $4a + 2b + c = 30$, $16a + 4b + c = 34$.
- Solve, by hand or on the calculator: $a = -0.125$, $b = 2.75$, $c = 25$.
- $\frac{D(4) - D(0)}{4 - 0} = \frac{34 - 25}{4} = 2.25$ thousand plays per month.
From $t = 0$ to $t = 4$, total plays increased by an average of 2.25 thousand per month.
- Start the line at an endpoint, $(0, 25)$: the estimate is $25 + 2.25(1.5) = 28.375$. The model gives $D(1.5) \approx 28.844$.

D is concave down on $(0, 4)$, so the secant lies below the graph and 28.375 is an underestimate.



Worked example

Light intensity is 90 units at 2 m from a lamp and follows $I = \frac{k}{d^2}$. What is it at 3 m?

1 $90 = \frac{k}{2^2}$, so $k = 360$.

$I(3) = \frac{360}{3^2} = 40$ units

Unit 2 Exponential and Logarithmic Functions

This unit is about change by multiplying instead of adding. Sequences lead to exponential functions and models; composition and inverse functions lead to logarithms, their rules, exact equations, and semi-log plots.

On the exam: 25–40% of the multiple-choice section, plus free-response questions 1 and 2 (calculator allowed) and question 4 (no calculator). Questions 1 and 2 use these functions and models; question 4 asks you to solve their equations exactly.

2.1 Arithmetic and geometric sequences

A sequence is a function whose inputs are whole numbers, so its graph is a row of separate dots, not a curve. An arithmetic sequence adds the same number d at every step. A geometric sequence multiplies by the same ratio r at every step.

Formulas

Arithmetic $a_n = a_0 + dn$, or from any known term, $a_n = a_k + d(n - k)$

Geometric $g_n = g_0 r^n$, or from any known term, $g_n = g_k r^{n-k}$

Worked example

A geometric sequence has $g_3 = 40$ and $g_6 = 320$. Write a formula for g_n .

- Three steps take 40 to 320, so $r^3 = \frac{320}{40} = 8$ and $r = 2$.
- $g_n = 40 \cdot 2^{n-3}$. Going back three steps, $g_0 = 40 \div 8 = 5$.

$g_n = 5 \cdot 2^n$ (the same sequence as $40 \cdot 2^{n-3}$)

Common mistake

Starting the count in the wrong place. If the first term is called a_1 , then $a_n = a_1 + (n - 1)d$, because the 1st term has had no steps yet. One cent on day 1, doubled every day, is 2^{29} cents on day 30, not 2^{30} .

2.2 Change in linear and exponential functions

Over equal-length input intervals, a linear function's outputs change by the same amount, and an exponential function's outputs change by the same factor. A linear function is built by repeated adding; an exponential function by repeated multiplying. If the inputs go up by 2, the ratio you see is b^2 : a table that halves every 2 units has $b = 0.5^{1/2} \approx 0.707$.

The function through two points

Linear $y = y_1 + m(x - x_1)$, with $m = \frac{y_2 - y_1}{x_2 - x_1}$

Exponential $y = y_1 b^{x-x_1}$, with $b = \left(\frac{y_2}{y_1}\right)^{1/(x_2-x_1)}$ ($x_1 \neq x_2$, $y_1 y_2 > 0$; $b = 1$ gives a constant).

Write it like this

For a table with $x = 0, 1, 2, 3$ and $y = 80, 60, 45, 33.75$: the inputs increase in equal steps of 1, and the outputs change by a constant factor, $\frac{60}{80} = \frac{45}{60} = \frac{33.75}{45} = 0.75$, so an exponential model is appropriate.

x	0	1	2	3	4	
Linear	5	8	11	14	17	add 3 each step
Exponential	5	10	20	40	80	multiply by 2 each step

Common mistake

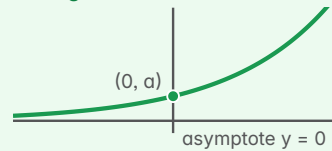
Giving "it decreases" as the reason a model is exponential.

Linear and exponential functions can both decrease. Name the equal input steps and the constant ratio, with numbers from the table.

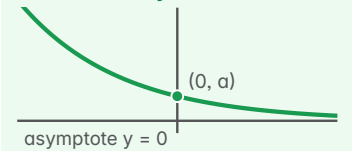
2.3 Exponential functions

An exponential function is $f(x) = ab^x$, with a not zero, $b > 0$ and $b \neq 1$. Here a is the value at $x = 0$ (the initial value) and b is the base. When $a > 0$, the range is $y > 0$, the x -axis is a horizontal asymptote, and the graph is always concave up. There are no turning points and no points of inflection. Adding k , as in $ab^x + k$, moves the asymptote to $y = k$.

$b > 1$: growth $ab^x, a > 0$



$0 < b < 1$: decay $ab^x, a > 0$



Common mistake

Calling a falling exponential concave down. $3(0.5)^x$ decreases, but its graph bends upward: it is concave up, because its drops get smaller. With $a > 0$, every exponential is concave up: it rises if $b > 1$ and falls if $0 < b < 1$. A negative a flips the graph upside down, so a rising curve falls and it becomes concave down.

2.4 Exponential function manipulation

Exponent rules, for $b > 0$

Product $b^m \cdot b^n = b^{m+n}$

Negative $b^{-n} = \frac{1}{b^n}$

Quotient $\frac{b^m}{b^n} = b^{m-n}$

Fraction $b^{m/k} = (\sqrt[k]{b})^m$

Power $(b^m)^n = b^{mn}$

Zero $b^0 = 1$

Worked example

Without a calculator, write $\frac{9^{x-1} \cdot 3^{2x}}{27}$ as a single power of 3.

- Use base 3 throughout: $9^{x-1} = 3^{2x-2}$ and $27 = 3^3$.
- Add the exponents on top, subtract the one below: $(2x - 2) + 2x - 3$.

3^{4x-5}

The rules explain graph changes too. Since $b^{x+k} = b^k \cdot b^x$, a sideways shift of an exponential is a vertical stretch or compression. Since $b^{cx} = (b^c)^x$, stretching or squeezing it sideways is a change of base: $2^{3x} = 8^x$. Fractional powers work the same way: $27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$.

⚠ Common mistake

Using the rules on sums. The exponent-addition rule applies to products, not sums. $2^x + 2^x = 2 \cdot 2^x = 2^{x+1}$, not 2^{2x} . And $3 \cdot 2^x$ is not 6^x : the 3 is not part of the base.

2.5 Exponential function context and data modeling

Growth of r per unit, written as a decimal, means a factor of $b = 1 + r$: 4% growth is 1.04 and a 15% loss is 0.85. Something that doubles every T units is $Q_0 \cdot 2^{t/T}$, and something that halves every H units is $Q_0 \left(\frac{1}{2}\right)^{t/H}$. The number $e \approx 2.718$ gives one more form: $ae^{kt} = a(e^k)^t$, so $b = e^k$.

📊 Worked example

A car is worth 27.2 thousand dollars at $t = 1$ year and 14.8 thousand at $t = 6$. Find a and b in $V(t) = ab^t$.

- Two equations: $ab = 27.2$ and $ab^6 = 14.8$.
- Divide them to cancel a : $b^5 = \frac{14.8}{27.2}$, so $b = 0.885398 \dots$ (store it; the keys are on page 1).
- $a = 27.2 \div b = 30.7206 \dots$

$a \approx 30.721$ and $b \approx 0.885$: the car loses about 11.5% of its value each year.

⚠ Common mistake

Treating the first data point as $t = 0$. Here the first value is at $t = 1$, so $27.2 = ab$, not a . Write each equation with the input the data actually give.

⚠ Common mistake

Rounding before the end. Dividing by the rounded $b = 0.885$ gives $a = 30.734$, wrong in the second decimal place. Keep b stored and divide by the stored value.

📱 Fit an exponential to data

TI-84 Plus CE. **[stat]** 1: Edit. Enter the inputs in L1 and the outputs in L2. Then **[stat]** → **CALC** → 0: ExpReg, set Xlist L1 and Ylist L2, and choose Calculate. It gives $y = a \cdot b^x$.

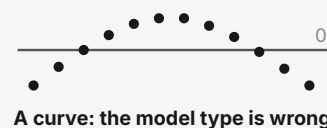
TI-Nspire CX. On a Lists & Spreadsheet page, enter the data in columns named x and y . **[menu]** → Statistics → Stat Calculations → Exponential Regression, with X List x and Y List y .

Desmos. Add a table (+ → table) with columns x_1 and y_1 , then type $y_1 \sim a \cdot b^{(x_1)}$.

2.6 Competing function model validation

which model fits better

A residual is the actual value minus the value the model predicts. A positive residual means the model underestimated; a negative one means it overestimated. Plot the residuals against the input. If they scatter with no pattern, the model's type fits the data; if they form a curve, it doesn't.



⚠ Common mistake

Trusting a high correlation. Stopping distances grow like a quadratic, yet a straight line through them can still have a correlation of $r \approx 0.99$ (close to 1, a nearly perfect line). Its residuals then curve, like the right-hand residual plot for this topic, which shows the line has the wrong shape. Check the residual plot, and check that the model makes sense in the situation.

2.7 Composition of functions

$f(g(x))$ means apply g first, then apply f to the result. Its domain is the inputs of g whose outputs f accepts. Order matters: $f(g(x))$ and $g(f(x))$ are usually different functions.

📊 Worked example

f is given only by this table, and $g(x) = 2^x$. Find $g(f(1))$, $f(g(1))$ and $f(f(3))$.

x	0	1	2	3	4
$f(x)$	5	3	0	-2	1

1 $g(f(1)) = g(3) = 2^3 = 8$

2 $f(g(1)) = f(2) = 0$, a different answer.

3 $f(f(3)) = f(-2)$, but the table has no input -2 .

8, 0, and "not defined"

✍ Write it like this

Show the inner value: $g(f(1)) = g(3) = 8$. On the exam, a correct final number without the middle step may not earn the point.

2.8 Inverse functions

If $f(a) = b$, then $f^{-1}(b) = a$: the inverse swaps inputs and outputs, so the domain of f^{-1} is the range of f . A function has an inverse only if no output comes from two different inputs; if one does, restrict the domain. The graph of f^{-1} is the graph of f reflected in the line $y = x$, and $f(f^{-1}(x)) = x$.

📊 Worked example

Find the inverse of $f(x) = (x - 1)^2 + 3$ for $x \geq 1$.

1 Swap x and y : $x = (y - 1)^2 + 3$, so $(y - 1)^2 = x - 3$.

2 On this branch $y \geq 1$, so take the positive root: $y = 1 + \sqrt{x - 3}$.

3 Check: $f(3) = 7$ and $f^{-1}(7) = 1 + 2 = 3$.

$f^{-1}(x) = 1 + \sqrt{x - 3}$, for $x \geq 3$

✍ Write it like this

$p(x) = x^2 + 1$ has no inverse on all real numbers, because $p(-2) = p(2) = 5$: the output 5 comes from two different inputs.

Point to this function's own values. "It fails the horizontal line test" on its own did not earn the point.

⚠ Common mistake

Reading f^{-1} as $1/f$. The -1 means "inverse", not "reciprocal". In the table above, $f^{-1}(3) = 1$, because $f(1) = 3$. The reciprocal $\frac{1}{f(3)} = \frac{1}{-2}$ is a different number.

2.9 Logarithmic expressions

A logarithm is an exponent. $\log_b c = a$ means $b^a = c$, for $b > 0$, $b \neq 1$ and $c > 0$. $\log x$ means base 10 and $\ln x$ means base e . For every base, $\log_b 1 = 0$ and $\log_b b = 1$.

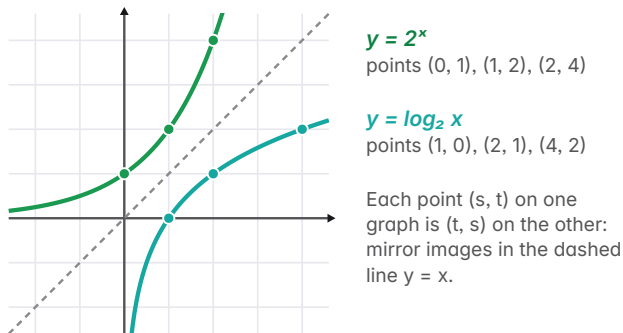
$$\begin{array}{ll} \log_2 32 = 5, \text{ because } 2^5 = 32 & \log_4 8 = \frac{3}{2}, \text{ because } 4^{3/2} = 8 \\ \log_3 \frac{1}{9} = -2, \text{ because } 3^{-2} = \frac{1}{9} & \ln \frac{1}{e} = -1, \text{ because } e^{-1} = \frac{1}{e} \\ \log 0.001 = -3, \text{ because } 10^{-3} = 0.001 & \log_5 \sqrt{5} = \frac{1}{2}, \text{ because } 5^{1/2} = \sqrt{5} \end{array}$$

⚠ Common mistake

Thinking a log can't be negative. Only the input of a log must be positive: $\log 0$ and $\log(-5)$ do not exist. The output can be any number, including a negative one: $\log 0.01 = -2$, because $10^{-2} = 0.01$.

2.10 Inverses of exponential functions

$y = \log_b x$ is the inverse of $y = b^x$. Its table is the exponential's table with the rows swapped, and its graph is the exponential's graph reflected in $y = x$. So the log has domain $x > 0$, range all real numbers, and the vertical asymptote $x = 0$.

**📖 Formulas**

Undoing $\log_b(b^x) = x$ for every real x , and $b^{\log_b x} = x$ for $x > 0$

Change of base $\log_b x = \frac{\ln x}{\ln b}$, so on a calculator $\log_2 20 = \frac{\ln 20}{\ln 2} \approx 4.322$

2.11 Logarithmic functions

$f(x) = a \log_b(x - h) + k$, with $a \neq 0$, $b > 0$ and $b \neq 1$, needs $x - h > 0$, so its domain is $x > h$ and its vertical asymptote is $x = h$. Its range is all real numbers and it has no horizontal asymptote. With $a > 0$ and $b > 1$ it increases and is concave down. When the positive log inputs $x - h$ multiply by a constant factor, the outputs change by a constant amount: $x = 1, 3, 9, 27$ gives $2 + 3 \log_3 x = 2, 5, 8, 11$.

🔗 Write it like this

$g(x) = 3 + \ln(2x - 4)$ has domain $x > 2$. As x decreases toward 2, $\lim_{x \rightarrow 2^+} g(x) = -\infty$.

Approach from the side the domain allows: write $x \rightarrow 2^+$, not $x \rightarrow 2$ or $x \rightarrow 2^-$.

⚠ Common mistake

Expecting a log to level off. A log keeps growing without bound, just very slowly: $\ln x$ reaches 10 only at $x \approx 22,026$. It has a vertical asymptote and no horizontal one.

2.12 Logarithmic function manipulation**📖 Log rules (for $M, N > 0$)**

Product $\log_b(MN) = \log_b M + \log_b N$

Quotient $\log_b \frac{M}{N} = \log_b M - \log_b N$

Power $\log_b M^p = p \log_b M$

Watch the domain when you use the power rule: $\ln x^2$ is defined for every $x \neq 0$, but $2 \ln x$ only for $x > 0$.

The product rule explains a graph fact: for $k, x > 0$, $\log_b(kx) = \log_b k + \log_b x$, so stretching a log graph sideways is the same as shifting it up or down.

📖 Worked example

Without a calculator, write $2 \log_3 x - \log_3 9 + 1$ as a single logarithm.

- 1 Make every term a log: $2 \log_3 x = \log_3 x^2$ and $1 = \log_3 3$.
- 2 A sum of logs is the log of a product, and a difference is the log of a quotient: $\log_3 \frac{x^2 \cdot 3}{9}$.

$$\log_3 \frac{x^2}{3}, \text{ for } x > 0$$

⚠ Common mistake

Splitting a log of a sum. No rule breaks up $\log(x + 3)$: it is not $\log x + \log 3$, which equals $\log(3x)$. Leave $\log(x + 3)$ as it is; the rules only work on products, quotients and powers.

2.13 Exponential and logarithmic equations and inequalities

Isolate the exponential or the log, then switch forms: $b^x = c$ means $x = \log_b c$. Check every answer in the original equation, because a log's input must stay positive. In the no-calculator part, answers must be exact and worked out: $\ln 4$, not a decimal, and 9, not 3^2 .

🔗 Write it like this

Because $e^x > 0$ for every real x , $e^x = -3$ has no solution.

Inequalities work the same way. With a base above 1 the direction stays: $2^x > 10$ means $x > \log_2 10$. With a base between 0 and 1 it flips: $(\frac{1}{2})^x > 4$ means $x < -2$. A log inequality also keeps the input positive: $\log_3(x - 1) \leq 2$ means $0 < x - 1 \leq 9$, so $1 < x \leq 10$.

In the calculator part, solve equations like $\ln(6x - 6) - 4.792 = -1.5$ by graphing both sides and finding where they meet, then give three decimals: $x \approx 5.483$.

Equation	Step	Answer
$2^{x+1} = 8^x$	same base: $x + 1 = 3x$	$x = \frac{1}{2}$
$3e^{2x} - 4 = 11$	$e^{2x} = 5$, so $2x = \ln 5$	$x = \frac{1}{2} \ln 5$
$5^x = 12$	log form	$x = \log_5 12 = \frac{\ln 12}{\ln 5}$
$\log_2(x + 3) = 4$	$x + 3 = 2^4$	$x = 13$
$\ln(2x + 1) = \ln(x + 4)$	equal logs, equal inputs: $2x + 1 = x + 4$	$x = 3$
$\log_2 x + \log_2(x - 2) = 3$	$x(x - 2) = 8$, so $x = 4$ or -2	$x = 4$; reject -2 , since $\log_2(-2)$ is undefined

Worked example

Without a calculator, solve $e^{2x} - e^x - 12 = 0$.

- It is a quadratic in e^x ; let $u = e^x$, so
 $u^2 - u - 12 = (u - 4)(u + 3) = 0$.
- $e^x = 4$ gives $x = \ln 4$. $e^x = -3$ has no solution.

$$x = \ln 4$$

Inverses of shifted functions ($a \neq 0$, $b > 0$, $b \neq 1$)

$$f(x) = ab^{x-h} + k \quad f^{-1}(x) = h + \log_b \frac{x-k}{a}$$

$$f(x) = a \log_b(x-h) + k \quad f^{-1}(x) = h + b^{(x-k)/a}$$

For example, $4 + 3e^{x-1}$ has inverse $1 + \ln \frac{x-4}{3}$, for $x > 4$. Check one point: $f(1) = 7$ and $f^{-1}(7) = 1 + \ln 1 = 1$.

Solve an equation by graphing

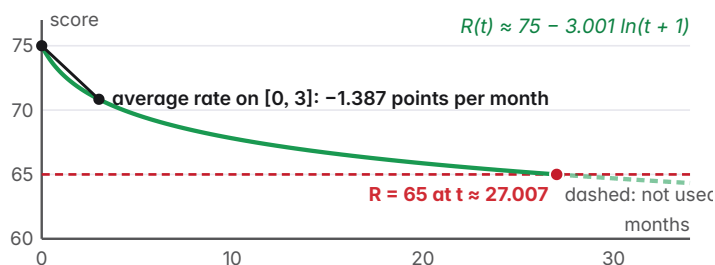
TI-84 Plus CE. Put the left side in Y1 and the right side in Y2. Press **2nd** **trace** (CALC), choose 5: intersect, press **enter** on the first curve, **enter** on the second, move near the crossing and press **enter**.

TI-Nspire CX. Graph both sides as f1 and f2. **menu** → Analyze Graph → Intersection, then click to the left of the crossing and to the right of it.

Desmos. Type each side as its own line, $y =$ left side and $y =$ right side. Tap where they cross to read the point.

2.14 Logarithmic function context and data modeling

A log model fits data whose positive log inputs (such as $t + 1$) multiply by a constant factor while the outputs change by a constant amount. Find its constants by putting data points into the given form, such as $a + b \ln(t + 1)$, or with a log regression on the calculator (LnReg on the TI-84, in the same STAT CALC menu as QuadReg). You won't be asked what a and b mean in context.

**Worked example**

A class averages 75 points on a test at $t = 0$ and 70.84 three months later. The model $R(t) = a + b \ln(t + 1)$ is used while $R \geq 65$. Find a and b , the average rate of change over the first 3 months, and the model's domain.

- $a + b \ln 1 = 75$, and $\ln 1 = 0$, so $a = 75$. Then
 $75 + b \ln 4 = 70.84$, so $b = \frac{-4.16}{\ln 4} \approx -3.001$.
- $\frac{70.84 - 75}{3 - 0} \approx -1.387$ points per month.
- $75 + b \ln(t + 1) = 65$ gives $t = e^{-10/b} - 1 \approx 27.007$.

The domain is $0 \leq t \leq 27.007$.

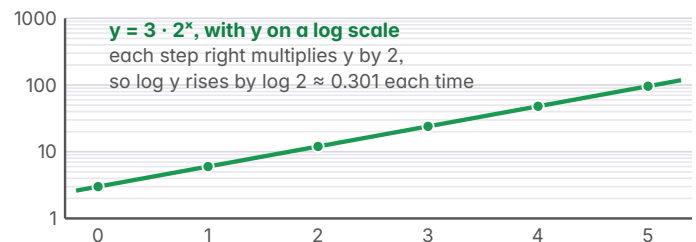
Write it like this

From $t = 0$ to $t = 3$ months, the score decreased by an average of 1.387 points per month.

The model is used until it predicts 65, which happens at $t \approx 27.007$, so its domain is $0 \leq t \leq 27.007$.

2.15 Semi-log plots

On a semi-log plot the vertical axis is logarithmic. Positive exponential data $y = ab^x$ then lie on a straight line, because $\log y = (\log b)x + \log a$: the slope is $\log b$ and the vertical intercept is $\log a$. Only the vertical axis is ever logarithmic on the exam.

**Worked example**

The line on a semi-log plot is $\log y = 2 + 0.5x$. Write y as an exponential function.

$$1 \quad y = 10^{2+0.5x} = 10^2 \cdot (10^{0.5})^x$$

$$y = 100 (10^{0.5})^x \approx 100(3.162)^x$$

Common mistake

Reading the slope as b . The slope is $\log b$. A slope of 0.5 on a base-10 log axis means $b = 10^{0.5} \approx 3.162$, not 0.5.

Which type of function fits a table (Units 1 and 2)

For the first four types, the inputs must go up in equal steps. Then look at the outputs:

Linear	the differences are constant
Quadratic	the second differences are constant and nonzero
Cubic	the third differences are constant and nonzero
Exponential	the ratios are constant, positive and not 1
Logarithmic	the inputs multiply by a constant while the outputs change by a constant amount

Give the reason with the table's own numbers: "the inputs go up by 1 and each output is 0.75 times the one before."

Unit 3

Trigonometric and Polar Functions

This unit is about functions that repeat. Angles measured in radians give sine, cosine and tangent on the unit circle. Their graphs become sinusoidal models, and inverse trig functions, equations and identities follow. Polar coordinates then describe points by angle and distance.

On the exam: 30–35% of the multiple-choice section, plus free-response questions 3 and 4, both without a calculator. Question 3 is always a periodic model built from a story; question 4 includes trig equations and identities.

3.1 Periodic phenomena

A periodic function repeats its outputs over equal-length input intervals. The period is the length of one full cycle: the smallest positive k with $f(x + k) = f(x)$ for every x . Its intervals of increase and decrease and its concavity repeat with the same period too.

Worked example

$f(t)$ for $t = 0, 1, 2, \dots, 8$ is 2, 5, 2, -1, 2, 5, 2, -1, 2, and the pattern continues. Find the period and $f(41)$.

- The values and their direction restart every 4 inputs: the period is 4.
- $41 = 4 \cdot 10 + 1$, so $f(41) = f(1)$.

Period 4, and $f(41) = 5$

Common mistake

Taking two equal heights as a full period. Above, $f(0) = f(2) = 2$, but the values rise through 2 at $t = 0$ and fall through it at $t = 2$. A period ends only where the value and the direction both repeat.

3.2 Sine, cosine, and tangent

An angle in standard position starts on the positive x -axis and turns counterclockwise when it is positive. If its terminal ray meets a circle of radius r at (x, y) , then $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, and, when $x \neq 0$, $\tan \theta = \frac{y}{x}$, the slope of the ray. One radian is the angle whose arc is as long as the radius.

Formulas

$|\theta| = \frac{s}{r}$, so the arc length is $s = r|\theta|$, with θ in radians. A full turn is 2π radians, or 360° : multiply degrees by $\frac{\pi}{180}$, so $45^\circ = \frac{\pi}{4}$.

Worked example

The terminal ray of θ passes through $(-3, 4)$. Find its sine, cosine and tangent.

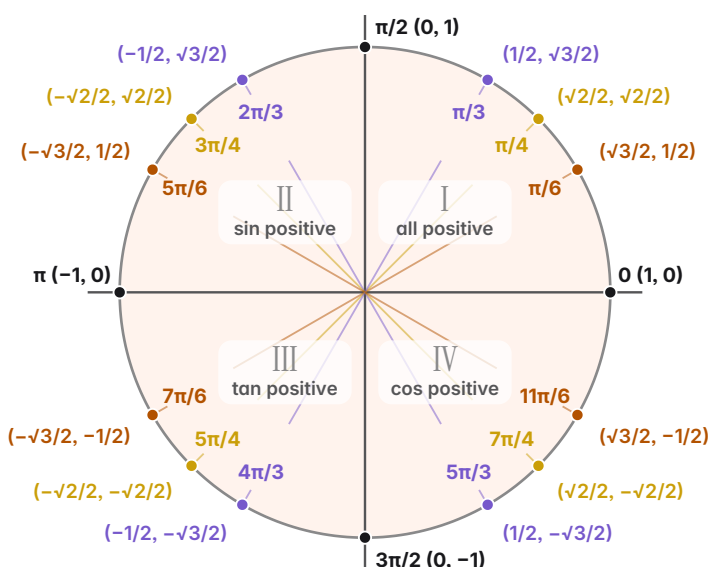
- $r = \sqrt{(-3)^2 + 4^2} = 5$, a distance, so never negative.

$$\sin \theta = \frac{4}{5}, \cos \theta = -\frac{3}{5}, \tan \theta = -\frac{4}{3}$$

Common mistake

Leaving the calculator in degree mode. $\cos 2$ means the cosine of 2 radians, -0.416 ; degree mode gives $\cos 2^\circ = 0.999$. Set radian mode first (keys on page 1).

3.3 Sine and cosine function values



Reference angle: ■ $\frac{\pi}{6}$ ■ $\frac{\pi}{4}$ ■ $\frac{\pi}{3}$ ■ on an axis

On the unit circle the point at angle θ is $(\cos \theta, \sin \theta)$. You need these values without a calculator.

Exact values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

At $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, sine runs $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$, and cosine runs the same list backward. Tangent is sine divided by cosine.

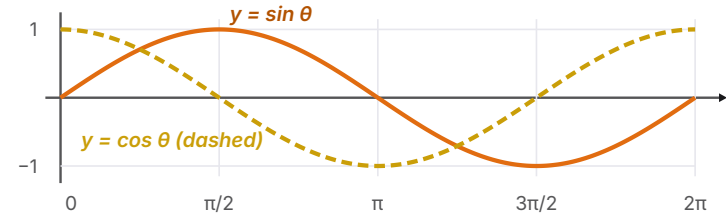
First reduce the angle to $0 \leq \theta < 2\pi$. Away from the axes, its reference angle is the acute angle to the x -axis: $\pi - \theta$ in quadrant II, $\theta - \pi$ in III, $2\pi - \theta$ in IV. Take its value from the table, then choose the sign from the quadrant. For example, $\sin \frac{7\pi}{6}$ has reference angle $\frac{\pi}{6}$ and lies in quadrant III, where sine is negative: $\sin \frac{7\pi}{6} = -\frac{1}{2}$.

Common mistake

Swapping the values at $\frac{\pi}{6}$ and $\frac{\pi}{3}$. $\sin \frac{\pi}{6} = \frac{1}{2}$, not $\frac{\sqrt{3}}{2}$. Picture the circle: the ray at $\frac{\pi}{6}$ is close to the x -axis, so its point is low (height $\frac{1}{2}$) and far out (across $\frac{\sqrt{3}}{2}$). At $\frac{\pi}{3}$ the two swap.

3.4 Sine and cosine function graphs

Plot the height of the point on the unit circle against the angle and you get $y = \sin \theta$; plot its horizontal position and you get $y = \cos \theta$. Both have domain all real numbers, range $[-1, 1]$ and period 2π . Sine starts at 0 and rises; cosine starts at its maximum. They are the same wave shifted: $\cos \theta = \sin(\theta + \frac{\pi}{2})$.

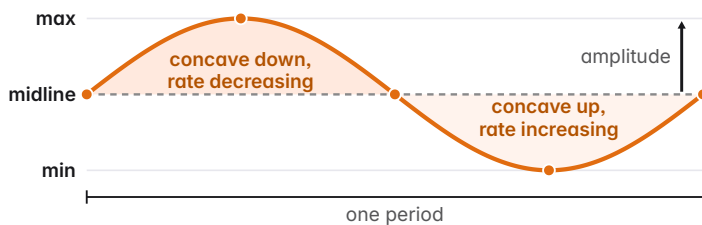


θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	1	0	-1	0
$\cos \theta$	1	0	-1	0	1

Sine is odd, $\sin(-\theta) = -\sin \theta$, and cosine is even, $\cos(-\theta) = \cos \theta$. So $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$ and $\cos(-\frac{\pi}{3}) = \frac{1}{2}$.

3.5 Sinusoidal functions

A sinusoid is any stretch or shift of the sine graph; cosine graphs count too. Its amplitude is $\frac{\max - \min}{2}$ and its midline is $y = \frac{\max + \min}{2}$. Its period is the input length of one cycle, and its frequency, $\frac{1}{\text{period}}$, is the number of cycles per unit of input. Above its midline a sinusoid is concave down, so its rate of change is decreasing; below the midline it is concave up and its rate of change is increasing.



Write it like this

If a sinusoid h has midline $y = 0$ and is below the x -axis and rising on (t_1, t_2) , write: "On (t_1, t_2) , h is negative and increasing. The graph of h is concave up, so the rate of change of h is increasing."

Start each claim with what it is about: h , the graph of h , or the rate of change of h . " h is increasing at a decreasing rate" is fine, but don't say the rate of change itself is "increasing at a decreasing rate": that describes how the rate's own rate changes, which needs calculus, and scorers have not accepted it.

Common mistake

Reading positive and negative from the midline. A function is positive where its graph is above the x -axis ($y = 0$), not above its midline. A tire point's height, $9 \sin(\pi(t - 1)) + 9$, is never negative, even on the lower half of its wave.

3.6 Sinusoidal function transformations

Four numbers move the sine graph, each in its own way.

The parts of $f(\theta) = a \sin(b(\theta + c)) + d$, with a and b not zero

Amplitude $|a|$; a negative a flips the wave over its midline

Period $\frac{2\pi}{|b|}$: a bigger $|b|$ means a shorter period

Phase shift $-c$: the graph moves c units left when $c > 0$

Midline $y = d$, and the range is $[d - |a|, d + |a|]$

Worked example

Describe $g(\theta) = -3 \sin(2\theta - \frac{\pi}{3}) + 4$.

1 Factor out b : $-3 \sin(2(\theta - \frac{\pi}{6})) + 4$.

2 Amplitude 3, period $\frac{2\pi}{2} = \pi$, midline $y = 4$, range $[1, 7]$.

The wave is shifted $\frac{\pi}{6}$ to the right, and because $a < 0$ it leaves the midline going down.

Common mistake

Reading the shift before factoring. In $\sin(2\theta - \frac{\pi}{3})$ the shift is $\frac{\pi}{6}$, not $\frac{\pi}{3}$, because $2\theta - \frac{\pi}{3} = 2(\theta - \frac{\pi}{6})$.

3.7 Sinusoidal function context and data modeling

To build a model from a story, get d and $|a|$ from the maximum and minimum, get $b = \frac{2\pi}{\text{period}}$ from the time between repeats, and set c from a known point and the direction of motion. For a sine with $a > 0$, the maxima occur where $b(t + c) = \frac{\pi}{2} + 2\pi k$, for integers k . Then check two more points a quarter period apart. If the question gives the sine form, answer in sine: a correct cosine model earns only part of the credit.

Worked example

High tide is 9 ft at 4 a.m., and the next low tide is 1 ft at 10 a.m. With t in hours after midnight, write $h(t) = a \sin(b(t + c)) + d$.

1 $d = \frac{9+1}{2} = 5$ and $a = \frac{9-1}{2} = 4$.

2 High to low is half a period, so the period is 12 h and $b = \frac{2\pi}{12} = \frac{\pi}{6}$.

3 Maximum at $t = 4$: $\frac{\pi}{6}(4 + c) = \frac{\pi}{2}$, so $c = -1$.

4 Check: $h(10) = 4 \sin(\frac{3\pi}{2}) + 5 = 1$.

$$h(t) = 4 \sin(\frac{\pi}{6}(t - 1)) + 5$$

Worked example in the style of question 3

A paddle wheel of radius 4 ft has its center 5 ft above the water and turns once every 6 seconds. At $t = 0$ a paddle tip is at its lowest point. Model the tip's height as $h(t) = a \sin(b(t + c)) + d$, and describe h on $(3, 4.5)$.

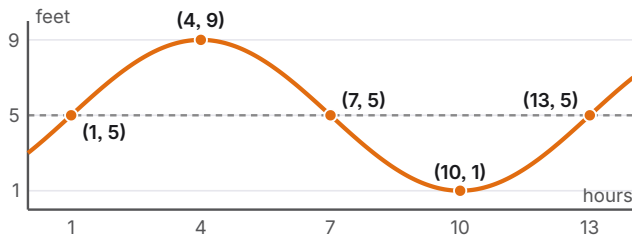
1 Part A's five points, a quarter period (1.5 s) apart: (0, 1) lowest, (1.5, 5), (3, 9) highest, (4.5, 5), (6, 1).

2 $a = 4$ (the radius), $d = 5$ (the center's height), $b = \frac{2\pi}{6} = \frac{\pi}{3}$.

3 A sine is lowest where $b(t + c) = -\frac{\pi}{2}$: $\frac{\pi}{3}(0 + c) = -\frac{\pi}{2}$, so $c = -\frac{3}{2}$.

4 On $(3, 4.5)$, h falls from 9 to 5: it is positive and decreasing. The graph is above its midline, so it is concave down and the rate of change of h is decreasing.

$$h(t) = 4 \sin(\frac{\pi}{3}(t - \frac{3}{2})) + 5. \text{ Check: } h(0) = 4 \sin(-\frac{\pi}{2}) + 5 = 1.$$



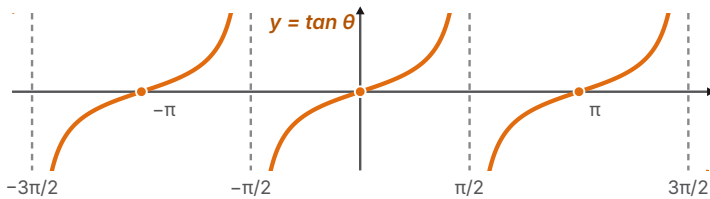
The tide model's points are a quarter period (3 hours) apart. Free-response question 3 asks for five coordinates like these.

Common mistake

Using the frequency as b . A string that vibrates 200 times a second has period $\frac{1}{200}$ s, so $b = \frac{2\pi}{\text{period}} = 2\pi \cdot 200 = 400\pi$, not 200.

3.8 The tangent function

$\tan \theta = \frac{\sin \theta}{\cos \theta}$ is the slope of the terminal ray. Its period is π , its zeros are at $\theta = k\pi$ for any integer k , and it has vertical asymptotes where $\cos \theta = 0$, at $\theta = \frac{\pi}{2} + k\pi$. It increases on every branch and takes every real value, so it has no amplitude or sinusoidal midline. For a $\tan(b(\theta + c)) + d$, with $a, b \neq 0$, the period is $\frac{\pi}{|b|}$.



Worked example

Find the period and the asymptotes of $y = 2 \tan(3x)$.

- The period is $\frac{\pi}{|b|} = \frac{\pi}{3}$. The asymptotes are where $3x = \frac{\pi}{2} + k\pi$.

Period $\frac{\pi}{3}$; asymptotes at $x = \frac{\pi}{6} + \frac{k\pi}{3}$. Check: at $x = \frac{\pi}{12}$, $y = 2 \tan \frac{\pi}{4} = 2$.

3.9 Inverse trigonometric functions

Sine, cosine and tangent repeat their values, so each is cut down to one stretch where every output appears once; that stretch becomes the inverse's range. The input of an inverse trig function is a value, and its output is an angle.



$\arcsin x$
from $-\pi/2$ to $\pi/2$



$\arccos x$
from 0 to π



$\arctan x$
between $-\pi/2$ and $\pi/2$

A filled dot is an angle the inverse can return; an open dot is one it never reaches.

x for arcsin, arccos	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
arcsin	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
arccos	π	$\frac{5\pi}{6}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	0
x for arctan	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
arctan	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

$\arcsin$ and $\arccos$ accept inputs from -1 to 1 ; $\arctan$ accepts any real number.

Worked example

Find $\arcsin(-\frac{1}{2})$, $\arccos(-\frac{1}{2})$, $\arctan(-\sqrt{3})$ and $\arcsin(\sin \frac{5\pi}{6})$.

- $\arcsin$ answers between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$: $-\frac{\pi}{6}$, not $\frac{7\pi}{6}$.
- $\arccos$ answers between 0 and π , so $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$, and $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$.
- $\sin \frac{5\pi}{6} = \frac{1}{2}$, and $\arcsin \frac{1}{2} = \frac{\pi}{6}$.

$$-\frac{\pi}{6}, \frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{6}$$

Worked example

Find $\sin(\arccos \frac{3}{5})$ without a calculator.

- $\theta = \arccos \frac{3}{5}$ has $\cos \theta = \frac{3}{5}$. Since $\frac{3}{5}$ is positive, θ is between 0 and $\frac{\pi}{2}$: draw a right triangle with adjacent side 3 and hypotenuse 5.
- The opposite side is $\sqrt{5^2 - 3^2} = 4$.

$$\sin \theta = \frac{4}{5}$$

Worked example

With a calculator in radian mode, solve $\sin x = 0.3$ for $0 \leq x < 2\pi$.

- $\arcsin 0.3 \approx 0.305$, an angle in quadrant I.
- Sine is also positive in quadrant II, at $\pi - \arcsin 0.3$; use the stored, unrounded angle.

$$x \approx 0.305 \text{ or } x \approx 2.837$$

Common mistake

Reading $\sin^{-1} x$ as $1/\sin x$. $\sin^{-1} x$ is arcsine, the angle whose sine is x . The reciprocal $1/\sin x$ is $\csc x$. The same goes for $\cos^{-1}$ and $\tan^{-1}$.

3.10 Trigonometric equations and inequalities

Isolate the trig function, find the reference angle, use every quadrant where the sign fits, then add whole periods. Keep exactly the answers inside the interval you're given. For a multiple angle such as $3x$, solve for $u = 3x$ over three times the interval, then divide by 3. A square root gives two signs: $\tan^2 x = 3$ means $\tan x = \pm\sqrt{3}$.

Worked example

Solve $2 \cos(2x) + 1 = 0$ for $0 \leq x < 2\pi$.

- $\cos(2x) = -\frac{1}{2}$, and $2x$ runs from 0 up to 4π : two turns.
- $2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$ (two answers per turn), then divide by 2.

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Common mistake

Dividing by a trig function. In $\sin x \cos x = \sin x$, dividing by $\sin x$ loses every x where $\sin x = 0$. Factor instead: $\sin x(\cos x - 1) = 0$, so $\sin x = 0$ or $\cos x = 1$. On $[0, 2\pi]$ that gives $x = 0$ and $x = \pi$; dividing would have kept only $x = 0$.

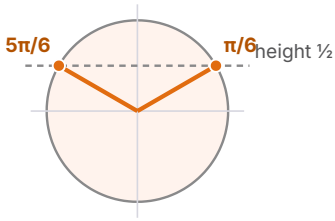
For an inequality, solve the equation to find the boundary points, then read the graph or test a point between them. $2 \cos \theta + 1 > 0$ on $[0, 2\pi]$ gives $[0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi]$.

Every solution of a trig equation ($\alpha = \arcsin c$, $\arccos c$ or $\arctan c$; k any integer)

$$\sin u = c \quad u = \alpha + 2\pi k \text{ or } u = \pi - \alpha + 2\pi k$$

$$\cos u = c \quad u = \alpha + 2\pi k \text{ or } u = -\alpha + 2\pi k$$

$$\tan u = c \quad u = \alpha + \pi k$$



$\sin u = 1/2$: the height $1/2$ meets the circle at two angles.

$$u = \pi/6 + 2\pi k$$

$$\text{or } u = \pi - \pi/6 + 2\pi k = 5\pi/6 + 2\pi k$$

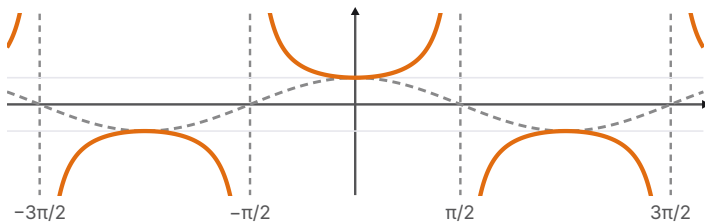
$\arcsin 1/2$ gives only $\pi/6$.

3.11 The secant, cosecant, and cotangent functions

Formulas

Definitions $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$, $\cot x = \frac{\cos x}{\sin x}$

Graphs Asymptotes where the bottom is 0. Secant and cosecant have period 2π and never take values between -1 and 1 ; cotangent has period π .



Solid: $y = \sec x$. Dashed: $y = \cos x$. Where $\cos x = 0$, $\sec x$ has an asymptote.

x	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sec x$	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2
$\csc x$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$
$\cot x$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$

Common mistake

Calling $\cot \frac{\pi}{2}$ undefined. $\cot \frac{\pi}{2} = \frac{\cos(\pi/2)}{\sin(\pi/2)} = \frac{0}{1} = 0$. It is undefined where $\sin x = 0$, at $x = k\pi$.

3.12 Equivalent representations of trigonometric functions

Pythagorean identities

$\sin^2 x + \cos^2 x = 1$. Divide it by $\cos^2 x$ to get $1 + \tan^2 x = \sec^2 x$, or by $\sin^2 x$ to get $1 + \cot^2 x = \csc^2 x$.

Worked example

Find $\cos \frac{\pi}{12}$ exactly.

1 $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$, so use the difference formula: $\cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

$$\cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4} \approx 0.966$$

Sum, difference and double-angle identities

Sum and difference $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$
 $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$

Double angle Set $\beta = \alpha$ above: $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$

Half-angle identities and the tangent sum and difference formulas are not on the exam.

Worked example

$\sin \theta = \frac{3}{5}$ and θ is in quadrant II. Find $\sin 2\theta$ and $\cos 2\theta$.

1 $\cos \theta = -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5}$, negative because cosine is negative in quadrant II.

2 $\sin 2\theta = 2 \left(\frac{3}{5}\right) \left(-\frac{4}{5}\right)$ and $\cos 2\theta = 1 - 2 \left(\frac{3}{5}\right)^2$.

$$\sin 2\theta = -\frac{24}{25} \text{ and } \cos 2\theta = \frac{7}{25}$$

Worked example

Solve $\cos 2x = \sin x$ for $0 \leq x < 2\pi$.

1 Choose the form with only sine: $1 - 2 \sin^2 x = \sin x$, so $2 \sin^2 x + \sin x - 1 = 0$.

2 $(2 \sin x - 1)(\sin x + 1) = 0$: $\sin x = \frac{1}{2}$ or $\sin x = -1$.

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

Worked example

Show that $\frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$ wherever both sides are defined.

1 Multiply the top and bottom of the left side by $1 + \cos x$: $\frac{\sin x(1 + \cos x)}{1 - \cos^2 x}$.

2 $1 - \cos^2 x = \sin^2 x$, so this is $\frac{\sin x(1 + \cos x)}{\sin^2 x}$.

Cancel one $\sin x$ to get $\frac{1 + \cos x}{\sin x}$.

Write it like this

To rewrite an expression in a requested form, show each step on its own line:

$$\frac{1 - \cos^2 x}{\sin x \cos x} = \frac{\sin^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x$$

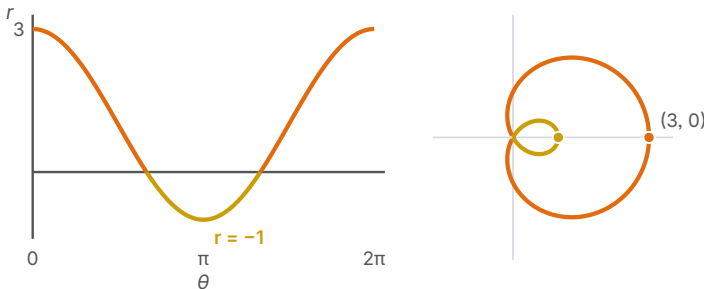
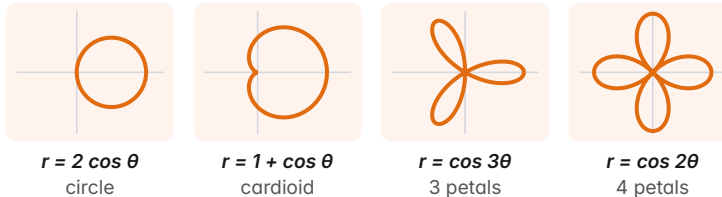
Stop at exactly the form asked for. The final expression alone does not earn the point.

Common mistake

Splitting a sine of a sum. $\sin(\alpha + \beta)$ is not $\sin \alpha + \sin \beta$. $\sin \frac{\pi}{4} + \sin \frac{\pi}{6} \approx 1.21$ is bigger than any sine can be, while the sum formula gives the true value, $\sin \frac{5\pi}{12} \approx 0.966$.

3.13 Trigonometry and polar coordinates

The polar point (r, θ) is reached by turning to angle θ and going a signed distance r along that ray from the pole (the origin). A negative r goes the same distance the opposite way, through the pole. One point has many names: (r, θ) , $(r, \theta + 2\pi)$ and $(-r, \theta + \pi)$ are all the same point.

Formulas**Polar to (x, y)** $x = r \cos \theta$, $y = r \sin \theta$ **(x, y) to polar** $r = \sqrt{x^2 + y^2}$; $\theta = \arctan \frac{y}{x}$ if $x > 0$, or $\arctan \frac{y}{x} + \pi$ if $x < 0$. If $x = 0$, use $\pi/2$ for $y > 0$ or $3\pi/2$ for $y < 0$; at the pole, any angle.**Complex numbers** $a + bi$ sits at (a, b) , so $a + bi = r(\cos \theta + i \sin \theta)$ **Worked example**Write the point $(-2, 2\sqrt{3})$ in polar form with $r > 0$.**1** $r = \sqrt{4 + 12} = 4$. Since $x < 0$,
 $\theta = \arctan(-\sqrt{3}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$. **$(4, \frac{2\pi}{3})$** **Common mistake****Using $\arctan \frac{y}{x}$ alone for a point on the left.** $\arctan$ only returns angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, which point into the right half of the plane. For $(-2, 2\sqrt{3})$ it gives $-\frac{\pi}{3}$, which points the opposite way; add π to get $\frac{2\pi}{3}$.**3.14 Polar function graphs**A polar function $r = f(\theta)$ takes an angle and gives a signed radius. To sketch it, find r at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and wherever $r = 0$, plot each point (the opposite way when r is negative), and join them in order of increasing θ . $r = 1 + 2 \cos \theta$: on the left, r against θ ; on the right, the curve it traces.
— $r > 0$ — $r < 0$, drawn on the opposite side of the poleThe interval of θ sets how much is drawn: $r = 4 \cos \theta$ traces its whole circle for $0 \leq \theta \leq \pi$, then goes round again from π to 2π .**Shapes to recognize (a and b not zero)** **$r = a \cos \theta$ or $a \sin \theta$** a circle through the pole **$r = a + b \cos \theta$ or $a + b \sin \theta$** an inner loop if $|a| < |b|$, a heart shape (cardioid) if $|a| = |b|$, no loop if $|a| > |b|$ **$r = a \cos(n\theta)$** for positive integers n , a rose with n petals if n is odd, $2n$ if n is even**Worked example**Sketch $r = 2 - 2 \sin \theta$ for $0 \leq \theta \leq 2\pi$.**1** At $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$: $r = 2, 0, 2, 4$. Here $|a| = |b| = 2$, so it is a cardioid.**A heart shape that touches the pole at $\theta = \frac{\pi}{2}$ and reaches farthest, 4 units, straight down at $\theta = \frac{3\pi}{2}$.****3.15 Rates of change in polar functions**The distance from the pole is $|r|$, so $r = -2$ is farther away than $r = 1$. Whether the point moves toward or away from the pole depends on the sign of r and on whether r is increasing, as in the table. The average rate of change of r on $[\alpha, \beta]$ is $\frac{r(\beta) - r(\alpha)}{\beta - \alpha}$, in units of radius per radian.

	r increasing	r decreasing
$r > 0$	moving away from the pole	moving toward the pole
$r < 0$	moving toward the pole	moving away from the pole

Write it like thisFor $r = 1 + 2 \cos \theta$ on $(\frac{2\pi}{3}, \pi)$: " r is negative and decreasing, so the distance between the point (r, θ) and the origin is increasing."**Worked example**For $r = 2 \cos \theta - 3$ on $[0, \frac{\pi}{2}]$, is the point moving toward or away from the pole?**1** r goes from -1 to -3 : negative and decreasing.**It moves away from the pole, because $|r|$ grows from 1 to 3, even though r decreases.****Worked example**For $r = 1 + 2 \cos \theta$, find the average rate of change of r on $[0, \frac{\pi}{2}]$ and use it to estimate $r(\frac{\pi}{4})$.**1** $\frac{r(\pi/2) - r(0)}{\pi/2 - 0} = \frac{1 - 3}{\pi/2} = -\frac{4}{\pi} \approx -1.273$ units of radius per radian.**2** $r(\frac{\pi}{4}) \approx 3 - \frac{4}{\pi} \cdot \frac{\pi}{4} = 2$.**The estimate is 2; the true value is $1 + \sqrt{2} \approx 2.414$.****Common mistake****Reading the rate of change of r as a slope.** $\frac{\Delta r}{\Delta \theta}$ measures the average change in signed radius per radian of turning; distance is $|r|$. It is not the slope of the curve in the xy -plane and not the speed along it.**Worked example** r changes linearly between these values. Where is the point getting closer to the pole?

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	3	1	-1	-2	-0.5

1 r is positive and decreasing until it hits 0 at $\frac{3\pi}{8}$, then negative and decreasing until $\frac{3\pi}{4}$, then negative and increasing.**Closer on $(0, \frac{3\pi}{8})$ and $(\frac{3\pi}{4}, \pi)$; farther in between**

Unit 4 Functions Involving Parameters, Vectors, and Matrices

This unit covers parametric functions, implicitly defined curves and conic sections, vectors, and matrices used as functions.

Not on the AP exam. The exam covers Units 1–3 only. Your teacher may still test Unit 4 in class, and AP Classroom has optional progress checks for it.

4.1 4.2 4.3 Parametric functions and motion

A parametric function gives x and y as separate functions of a parameter t , often time, so the point $(x(t), y(t))$ traces a path as t increases. The point moves right while $x(t)$ increases and up while $y(t)$ increases, so if $x(t)$ is increasing and $y(t)$ decreasing at $t = 2$, it is moving right and down. Over an interval, the average rates of change are $\frac{\Delta x}{\Delta t}$ and $\frac{\Delta y}{\Delta t}$, and when $\Delta x \neq 0$, their ratio $\frac{\Delta y}{\Delta x}$ is the slope of the segment between the two points.

4.4 Parametrically defined circles and lines

Formulas

Circle $(h + r \cos t, k + r \sin t)$ for $0 \leq t \leq 2\pi$: center (h, k) , radius r , once counterclockwise from the rightmost point

Segment from P to Q : $P + t(Q - P)$ for $0 \leq t \leq 1$

4.5 4.6 4.7 Implicit curves and conic sections

An equation in x and y , such as $x^2 + y^2 = 25$, can describe a curve that is not one function; solving for y splits it into pieces.

Formulas

Ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
traced by $(h + a \cos t, k + b \sin t)$

Hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
asymptotes $y - k = \pm \frac{b}{a}(x - h)$

Parabola $y - k = A(x - h)^2$ opens up or down
 $x - h = A(y - k)^2$ opens sideways

Worked example

Find the center and the horizontal and vertical radii of $x^2 + 4y^2 - 6x = 7$.

1 Add 9 to both sides to complete the square:
 $(x - 3)^2 + 4y^2 = 16$. Then divide by 16.

$\frac{(x-3)^2}{16} + \frac{y^2}{4} = 1$, so the center is $(3, 0)$, the horizontal radius 4 and the vertical radius 2.

4.8 4.9 Vectors and vector-valued functions

The vector from P to Q is $Q - P$ (head minus tail). A vector-valued function $p(t) = \langle x(t), y(t) \rangle$ points from the origin to the moving point.

Formulas, for nonzero $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$

Length $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$, so $\langle 3, 4 \rangle$ has length 5. A unit vector in the same direction is $\frac{\vec{u}}{\|\vec{u}\|}$.

Dot product $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 = \|\vec{u}\|\|\vec{v}\|\cos\theta$; it is 0 exactly when the vectors are perpendicular

4.10 4.11 Matrices, inverses, and determinants

A product AB multiplies rows of A by columns of B , so an $m \times n$ matrix times an $n \times p$ matrix gives an $m \times p$ matrix.

Formulas

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

Determinant $\det A = ad - bc$

Inverse $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, which exists only when $ad - bc$ is not 0

Worked example

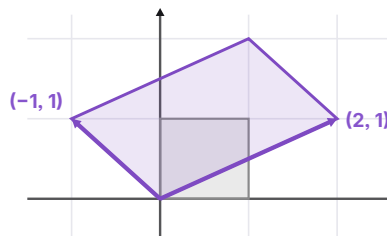
Find the inverse of $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

1 $\det A = 2 \cdot 3 - 1 \cdot 5 = 1$, which is not 0, so A has an inverse.

$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$. Check: AA^{-1} is the identity matrix.

4.12 4.13 Linear transformations and matrices

A 2×2 matrix A acts on points as a function, $\vec{v} \mapsto A\vec{v}$. Its columns are where $(1, 0)$ and $(0, 1)$ land; in the picture, $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. $|\det A|$ is the factor by which every area changes. In BA , the transformation A happens first. A counterclockwise rotation by θ is $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.



A sends $(1, 0)$ to $(2, 1)$, its first column, and $(0, 1)$ to $(-1, 1)$, its second.

The grey square (area 1) becomes the purple shape, with area $|\det A| = 3$.

4.14 Matrices modeling contexts

In a two-state model, each column of the transition matrix M shows where one state's members go in one step. From a starting state $\vec{s}_0$, the state after n steps is $M^n \vec{s}_0$; if M is invertible, the step before $\vec{s}$ is $M^{-1}\vec{s}$, and a steady state satisfies $M\vec{s} = \vec{s}$.

The free-response questions

Four questions, 6 points each. Every question has three parts, and each part has two points. Questions 1 and 2 come first, with a calculator, in 35 minutes; questions 3 and 4 follow, without one, in another 35 minutes.

New for May 2027: question 2's part B includes interpreting an average rate of change. In the CED sample, part C asks for an equation and a domain restriction.

Question 1 Function concepts (calculator)

Functions come as a graph, a table and a formula, and the parts mix them.

Part A Values such as a composition, $g(f(2))$, and every input that gives a certain output. Show the middle value: if the graph shows $f(2) = 5$, write " $g(f(2)) = g(5)$ ", then the value of $g(5)$ from its formula. A correct decimal with no middle step may not earn the point.

Part B Solve an equation on the calculator (every solution, to three decimals), and describe end or asymptote behavior as a limit with all four pieces: lim, the approach, the function and the value.

Part C A question about the functions, such as whether one is invertible or which type of function fits a table, and the reason. Base the reason on this function: its values, its points, or a property such as "f is increasing on its whole domain". "It passes the horizontal line test" alone did not earn the point.

Write it like this

Yes, f is invertible. f is increasing on its whole domain, so no two points on its graph have the same y -coordinate: each output comes from exactly one input.

Question 2 Modeling a non-periodic context (calculator)

A real situation with data and a model in a given form: polynomial, piecewise, exponential or logarithmic. Topic 2.14 works one through from start to finish.

Part A One equation for each data point, written out, then the constants as decimals. The calculator can solve the equations, but the equations themselves earn the first point.

Part B An average rate of change, shown as a quotient with the data values, then a sentence that interprets it: the interval, "on average", up or down, and the units.

Part C A question about the model in context, then a justified conclusion about its assumptions or limitations. In the CED sample, write and solve an equation for a target value, then explain a domain restriction using the situation.

Sentences to fill in

From $t = \underline{\text{start}}$ to $t = \underline{\text{end}}$ units, the quantity increased or decreased by an average of rate units per unit of time.

The model only makes sense for interval, because a fact about the situation, but the model what it does outside that interval.

Question 3 Modeling a periodic context (no calculator)

A sinusoid models something that repeats, like a turning wheel or a tide. Topic 3.7 works one through.

Part A The coordinates of five labeled points on the graph, a quarter period apart: heights from the maximum, midline and minimum, and times from the period and the starting point.

Part B The constants in the given form $a \sin(b(t + c)) + d$ or its cosine version: one scoring point for a and d , another for b and c . If the question gives the sine form, answer in sine.

Part C On a labeled interval, say whether the function is positive or negative and whether it is increasing or decreasing. Then say whether the graph is concave up or down and whether the rate of change is increasing or decreasing.

Where points are lost

After the 2025 exam, the scorers named two weak spots: turning the period and starting point into b and c (use $b = \frac{2\pi}{\text{period}}$, where the period is 1 over the frequency), and stating both the concavity and the rate of change in part C.

Question 4 Symbolic manipulations (no calculator)

The functions are exponential, logarithmic, trigonometric or inverse trig, and you work by hand.

Two of the parts Solve equations exactly, showing the steps. Reject impossible values, such as $e^x = -3$ (e^x is always positive) or an x that makes a log's input zero or negative, and give every solution in the interval, or the general solution when it is asked for.

The other part Rewrite expressions in a requested form, such as a single logarithm, a single power, or one trig function, with every step shown.

What every answer needs

- Show the work a part asks for; a correct answer alone may earn nothing.
- Give decimals accurate to three places, rounded or truncated; trailing zeros may be omitted. In the 2026 rubrics, the first answer with too few digits, or left exact when a decimal was requested, lost its point. Later such answers in that question were eligible for credit if otherwise correct. Follow the requested form on your exam.
- In questions 3 and 4, give exact values worked out as far as they go: $\frac{\pi}{6}$, not $\cos^{-1} \frac{\sqrt{3}}{2}$ or 0.524.
- Don't add claims you haven't checked: an extra wrong limit or solution can cancel a right one.