

Unit 1

AP Precalculus

Polynomial and Rational Functions

Rates of change · Polynomials · Rational functions · Asymptotes · Transformations · Modeling

Unit 1 30–40%
Polynomial and rational functions

Unit 2 25–40%
Exponential and logarithmic functions

Unit 3 30–35%
Trigonometric and polar functions

Unit 4 not tested
Parameters, vectors and matrices

Unit 1 is 30–40% of the multiple-choice section. Free-response questions 1 and 2 both use it, and both allow a graphing calculator. Question 1 asks for things like zeros, end behavior or the model a table fits; question 2 builds a model from data and uses it.

1.1
Change in tandem

1.2
Rates of change

1.3
Linear and quadratic rates

1.4
Polynomials and rates

1.5
Complex zeros

1.6
Polynomial end behavior

1.7
Rational end behavior

1.8
Zeros of rational functions

1.9
Vertical asymptotes

1.10
Holes

1.11
Equivalent forms

1.12
Transformations

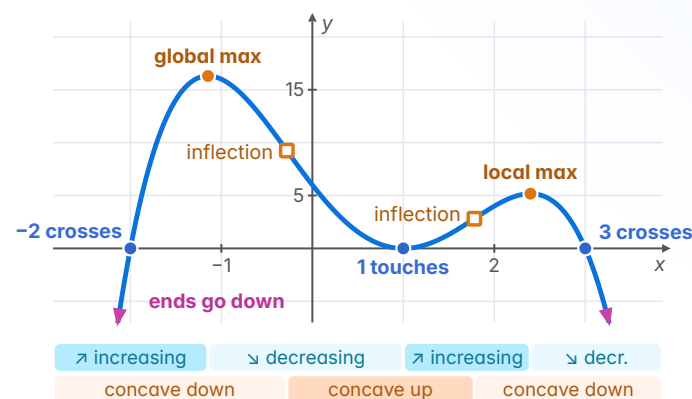
1.13
Choosing a model

1.14
Building a model

Polynomial

$$p(x) = -(x+2)(x-1)^2(x-3)$$

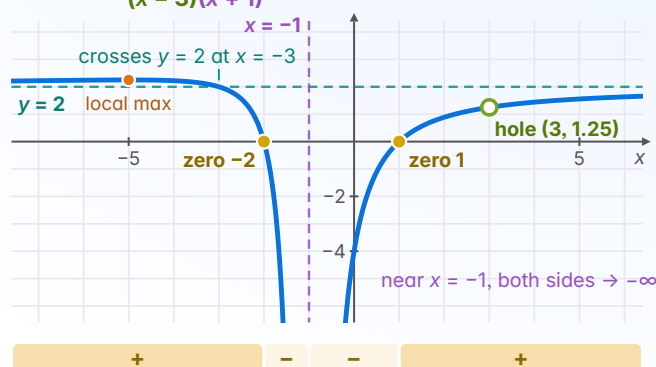
degree 4, leading term $-x^4$



Rational

$$R(x) = \frac{2(x+2)(x-1)(x-3)}{(x-3)(x+1)^2}$$

end behavior: $2x^3/x^3 = 2$



Sign changes only at the zeros -2 and 1 : $(x+1)^2$ is never negative.

The zeros, hole, asymptotes and ends come straight from the factors and leading terms; the turning and inflection points need a calculator.

What the exam asks most

- An average rate of change, with the work shown.** If a song had 25 thousand plays at month 0 and 34 thousand at month 4, write $(34 - 25)/(4 - 0) = 2.25$ and give the units: thousand plays per month. **1.2**
- A model built from data.** Write one equation for each data point, such as $a(2)^2 + b(2) + c = 30$, then solve them on your calculator (page 10 shows how). Give decimals. **1.14**
- End behavior written as a limit.** "As x increases without bound, $g(x)$ decreases without bound" is $\lim_{x \rightarrow \infty} g(x) = -\infty$, with all four parts: $\lim$, $x \rightarrow \infty$, g , the value. **1.6** **1.7**
- Every zero, to three decimal places.** Zoom out so no turn is hidden, then use the zero tool, or the minimum or maximum tool where the graph only touches the axis. Missing one loses the point. **1.5**
- Which model fits a table, and why.** If the inputs go up in equal steps and the second differences are constant, the data are quadratic. Say so and quote the differences. **1.3** **1.13**
- Where a model stops making sense.** Give a fact about the context and show the model breaking it: a song's total plays can't go down, but the model falls after month 11. **1.13**

Not on the exam

The rational root theorem, Descartes' rule of signs, synthetic division, partial fractions.

Formulas to know by heart

The exam gives you no formula sheet.

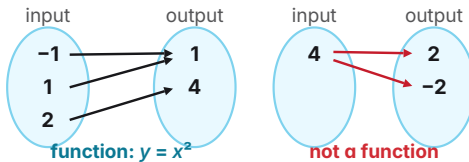
- Average rate of change** on $[a, b]$: $\frac{f(b) - f(a)}{b - a}$, in output units per input unit.
- Secant line** through $(a, f(a))$ with average rate m : $y = f(a) + m(x - a)$.
- Vertex** of $y = ax^2 + bx + c$ at $x = -b/(2a)$.
- Quadratic formula**: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Factor theorem**: $p(c) = 0$ exactly when $(x - c)$ is a factor.
- Division**: $p = d \cdot q + r$ (divisor d , quotient q , remainder r), where $r = 0$ or $\deg r < \deg d$.
- Transformations**: $Af(B(x - h)) + k$ moves each point (u, v) of the graph of f to $(h + u/B, Av + k)$.

1.1 Change in tandem

How the output changes as the input increases

Function words

Function	each input gives exactly one output
Domain, range	the set of inputs; the set of outputs
Image of a	the one output $f(a)$
Preimage of b	each x with $f(x) = b$ (often several)
Zero	an input with $f(x) = 0$, where the graph meets the x -axis

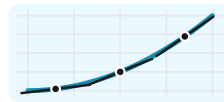


Two inputs can share an output, but one input can never have two outputs.

⚠ Exam trap

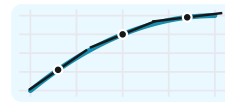
- Giving one answer to "find all x ". "Find all x with $f(x) = 1$ " asks for every input that gives 1. Draw the line $y = 1$ and read every crossing: if there are three, at $x = -2, 0.5$ and 3 , give all three.

concave up



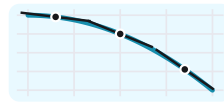
rising faster; slope rises

concave down



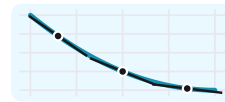
rising slower; slope falls

concave down



falling faster; slope falls

concave up



falling slower; slope rises

The short lines show the slope at each point. Concave up: the slopes keep increasing. Concave down: they keep decreasing. A falling curve has negative slopes, so "falling faster" means they get more negative.

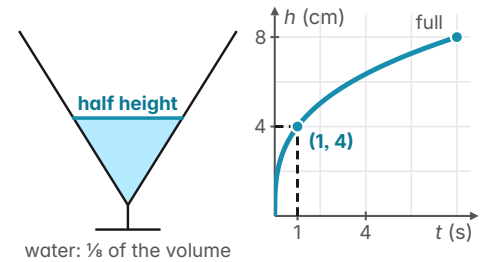
📖 Know by heart

Increasing on an interval: for every $a < b$ in it, $f(a) < f(b)$. **Decreasing**: $f(a) > f(b)$.

Concave up means the rate of change is increasing. **Concave down** means it is decreasing.

📊 Graphing a story

Label both axes with the quantity and its units.



Water pours at a steady rate into a cone-shaped glass 8 cm deep and fills it in 8 s. The water is **half as deep after 1 s** (half the depth holds only $\frac{1}{8}$ of the volume), then rises more and more slowly: $h(t) = 4t^{1/3}$ is increasing and concave down.

🔗 Describe how it changes

On the interval $0 < t < 8$, as t increases, h **increases at a decreasing rate**, so the graph is concave down.

To describe the rate itself, say only whether it goes up or down: "the rate of change of h is decreasing." How fast the rate changes is a calculus question.

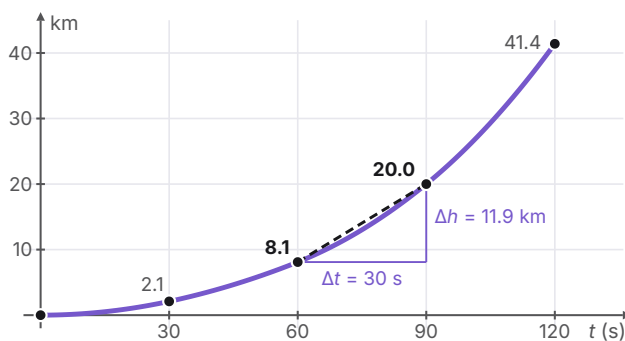
⚠ Exam trap

Mixing up positive and increasing.

Positive means above the axis; increasing means rising. $f(x) = x - 5$ on $0 < x < 5$ is negative but increasing.

1.2 Rates of change

The average rate of change is the slope of the secant line



Average rate
km per second

0 → 30 s	0.070
30 → 60 s	0.200
60 → 90 s	0.397
90 → 120 s	0.713

Each rate is bigger than the last, so it is concave up.

A rocket's altitude every 30 s. A rocket climbing at a steady 0.397 km/s would follow the dashed secant from (60, 8.1) to (90, 20.0): it covers the same 11.9 km in the same 30 s.

📊 Worked example

Average rate of change of altitude from $t = 60$ to $t = 90$ s, and its meaning.

$$1 \quad \frac{h(90) - h(60)}{90 - 60} = \frac{20.0 - 8.1}{30} = \frac{11.9}{30} \approx 0.397 \text{ km/s}$$

2 In words:

From $t = 60$ to $t = 90$ s, the altitude increased by 0.397 km per second on average.

🕒 The rate at one instant

Use the narrowest interval around that time the data allow. At $t = 60$: $(20.0 - 2.1)/(90 - 30) \approx 0.298$ km/s. Narrower intervals get closer to the rate at that instant.

Compare two instants the same way: near $t = 90$, $(41.4 - 8.1)/(120 - 60) = 0.555$ km/s, so the rocket climbs faster at 90 s than at 60 s.

📖 Know by heart

$$\frac{f(b) - f(a)}{b - a} \text{ for } a \neq b$$

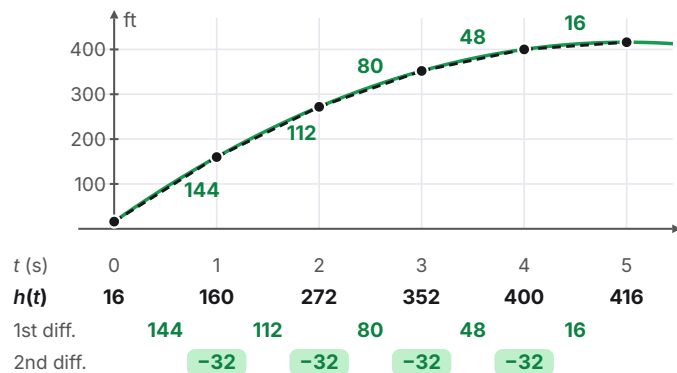
- Units: output units per input unit.
- Positive if the output ends higher than it started, negative if lower.
- Subtract in the same order on top and bottom.

⚠ Exam trap

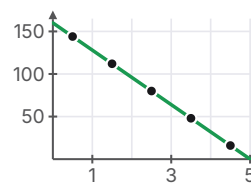
- **Dividing the height by the time.** For the rate at $t = 60$, $8.1/60 = 0.135$ km/s is only the average since launch; the rate near 60 s is about 0.298 km/s.
- **Writing "decreased by -2.7".** Fuel falls from 411 to 303 tonnes in 40 s, a rate of -2.7 tonnes per second. The minus sign already says "decreased": write "decreased by 2.7 tonnes per second on average".
- **Reading an average rate of 0 as "no change".** It only means the output ended where it started: a ball thrown up and caught at the same height has an average rate of 0.

1.3 Rates in linear and quadratic functions

A line's rate of change is constant; a quadratic's rate changes by the same amount each step



Average rate, ft/s



falls 32 ft/s each second: a straight line

Know by heart

- Linear, $mx + b$: the average rate of change is m on every interval.
- Quadratic: over equal, back-to-back intervals the average rates go up or down by the same amount each time.
- The second difference of $ax^2 + bx + c$, for inputs spaced Δx apart, is $2a(\Delta x)^2$. Cannonball: $2(-16)(1)^2 = -32$.

A cannonball, $h(t) = -16t^2 + 160t + 16$ feet. Each 1-second average rate (a dashed secant slope) is 32 ft/s less than the one before, so the second differences are constant. That is how you recognize a quadratic.

Worked example

Stopping distances at 30, 40, 50 and 60 mph are 75, 120, 175 and 240 ft. Which model fits, and why?

- Speeds rise in equal steps of 10 mph.
- First differences 45, 55, 65 change; second differences 10, 10 are constant.

Quadratic. (It is $d = 0.05v^2 + v$.)

Write the reason like this

The inputs increase in equal steps of 10, and the second differences of the outputs are constant (10, 10), so a quadratic model is best.

Quote the values from the table. "It looks curved" earns nothing, and neither does a high r^2 (a measure of how well a regression fits). Question 1 can end this way.

Exam trap

- Using differences when the inputs are unevenly spaced. Differences only compare fairly over equal steps. For $y = x^2$ at $x = 0, 1, 3, 4$ the differences 1, 8, 7 look random; divide each by its step to get rates 1, 4, 7, which show y rising faster and faster.
- Calling a quadratic's rate constant. For $y = x^2$ the rates over $[0, 1]$, $[1, 2]$, $[2, 3]$ are 1, 3, 5: they change, by 2 each time. Only a line's rate stays the same.

1.4 Polynomial functions and rates of change

Where a polynomial turns and bends, and its highest and lowest values

Know by heart

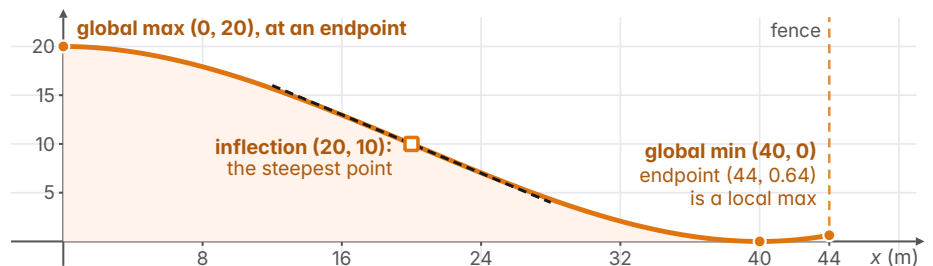
In $p(x) = a_n x^n + \dots + a_1 x + a_0$ with $a_n \neq 0$, n is the **degree**, $a_n x^n$ the **leading term** and a_n the **leading coefficient**.

Turning points, extremes and bends

- A **local maximum** or **minimum** is where the function switches between increasing and decreasing, or an endpoint higher or lower than the points next to it. A **global** one is the largest or smallest output of all.
- On all real numbers, an **even-degree** polynomial has a global maximum or minimum. An **odd-degree** one has neither: its ends go opposite ways, so it crosses the x -axis at least once.
- At an **inflection point** the rate of change switches between increasing and decreasing, so the concavity changes.

On the calculator

Zoom out so no turn hides off the screen, then use the maximum or minimum tool (keys on page 10).



8 m rates -0.26 -0.62 -0.74 -0.62 -0.26 falls, then rises

A snowboard landing, $L(x) = (x - 40)^2(x + 20)/1600$ on $0 \leq x \leq 44$. On $0 \leq x \leq 44$ the slope rises again after $x = 40$, so the endpoint (44, 0.64) is a local maximum. The steepest point, at $x = 20$, is the inflection point, where the 8 m rates stop falling and start rising.

Worked example

How high does the cannonball (above) go?

- Vertex at $t = -\frac{b}{2a} = \frac{160}{32} = 5$ s, and $h(5) = -400 + 800 + 16$.

Global maximum 416 ft, at $t = 5$ s

Cubic shortcut

A cubic's inflection point is halfway between its turning points. On all real numbers L turns at 0 and 40, so at 20.

Exam trap

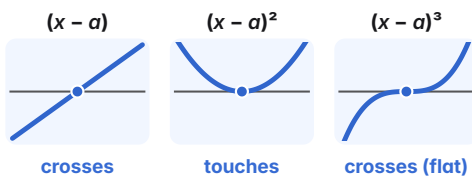
- Giving the input as the maximum. The maximum is the output, 416 ft; it happens at $t = 5$ s. "The maximum is 5" loses the point.
- Calling every flat spot an inflection point. $y = x^4$ is flat at 0 but concave up on both sides, so it has no inflection point there. Check that the rates really switch from falling to rising, or the other way.

1.5 Polynomial functions and complex zeros

Each zero a gives a factor $(x - a)$; degree n means n complex zeros

Know by heart

- If $p(a) = 0$ for a real number a , then $(x - a)$ is a factor and the graph meets the x -axis at $(a, 0)$.
- A polynomial of degree n has exactly n **complex** zeros, counting repeated ones.
- With real coefficients, non-real zeros come in **conjugate pairs**: $a + bi$ and $a - bi$.
- $i^2 = -1$, and the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives every zero of $ax^2 + bx + c$, real or not.

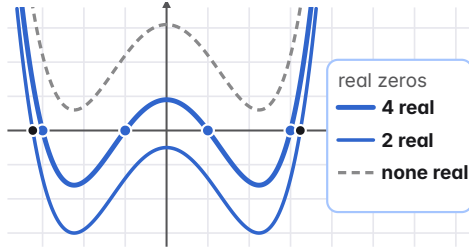


Odd power: the graph crosses the axis. Even power: it touches and turns back, keeping its sign.

Quick check

x : 0, 1, 2, 3, 4 and y : 0, 1, 8, 27, 64.
Least-degree polynomial model?

Answer: differences 1, 7, 19, 37, then 6, 12, 18, then 6, 6: the third ones are the first constant ones, so degree 3.



Adding a constant to $x^4 - 10x^2 + 9$ moves the graph up or down. It then has 4, 2 or 0 real zeros (counting repeats), never 3, because non-real zeros come in pairs.

Worked example

Least-degree polynomial with real coefficients and zeros 3 (multiplicity 2) and $2 + i$.

- $2 + i$ forces its conjugate $2 - i$.
- $(x - 2 - i)(x - 2 + i) = (x - 2)^2 - i^2 = x^2 - 4x + 5$

$$p(x) = (x - 3)^2(x^2 - 4x + 5), \text{ degree } 4$$

The factoring you do by hand

The exam only expects a common factor and quadratic patterns:

$$\begin{aligned} x^3 - 4x &= x(x - 2)(x + 2); \\ x^4 - 10x^2 + 9 &= (x^2 - 1)(x^2 - 9), \text{ then} \\ &= (x - 1)(x + 1)(x - 3)(x + 3). \end{aligned}$$

Polynomial inequalities



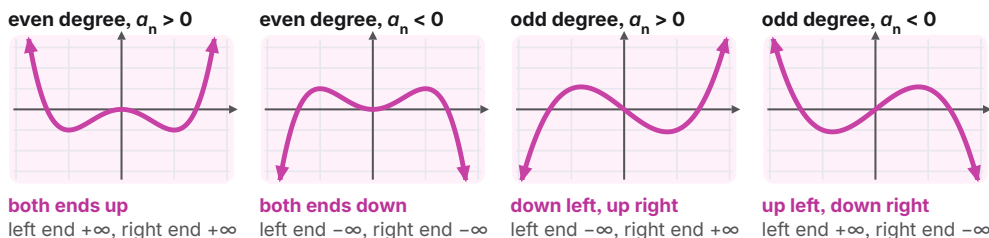
To solve $(x + 2)(x - 1)^2(x - 3) < 0$, test one input in every interval, the two outer ones too. The shaded answer is $(-2, 1) \cup (1, 3)$ ($\cup$ means "or"); 1 is left out because the product is 0 there. Use brackets only for $\leq$ or $\geq$.

Exam trap

- Confusing even degree with an even function.** An even function has $f(-x) = f(x)$. $x^2 + x$ has even degree, but $f(1) = 2$ and $f(-1) = 0$, so it is not even.
- Forgetting repeated zeros when you count.** $(x - 1)^2(x + 4)$ has degree 3 and three zeros, 1, 1 and -4 , though its graph meets the axis only twice.
- Treating non-real zeros as intercepts.** $x^2 + 1$ has zeros $\pm i$ but never meets the x -axis: only real zeros are intercepts.

1.6 Polynomial functions and end behavior

Far from zero, the leading term controls the graph



a_n is the leading coefficient. Even degree: both ends go the same way. Odd degree: opposite ways. A positive a_n sends the right end up.

Worked example

End behavior of $p(x) = (2x - 1)^3(4 - x)$, in limit notation.

- Leading term: $(2x)^3(-x) = -8x^4$.
- Even degree, negative a_n : both ends go down.

$$\lim_{x \rightarrow -\infty} p(x) = -\infty, \lim_{x \rightarrow \infty} p(x) = -\infty$$

How a free-response question asked it

For $g(x) = -0.167x^3 + x^2 - 1.834$: find every zero to three decimals, then the end behavior as x increases without bound.

Full credit: all three zeros, -1.233 , 1.578 and 5.643 , and $\lim_{x \rightarrow \infty} g(x) = -\infty$.

Why the highest power takes over

In $S(k) = 10k^2 - k^3$ the k^2 term is bigger at first ($S(6) = 144$), but k^3 takes over: $S(20) = -4,000$.

Read the equation off a graph

- The degree is even if both ends go the same way; $a_n > 0$ if the right end rises.
- A factor has an odd power where the graph crosses, an even power where it touches.
- One more point gives a : if $y = a(x + 1)^2(x - 2)$ passes through $(0, 4)$, then $4 = -2a$, so $a = -2$.

The four parts of a limit

$\lim_{x \rightarrow \infty} p(x) = \infty$: **lim**, where x goes ($x \rightarrow \infty$), the **function**, and what the output does.

Write it like this

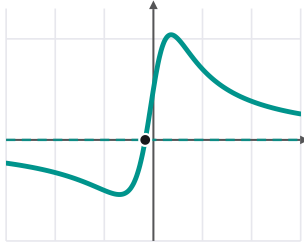
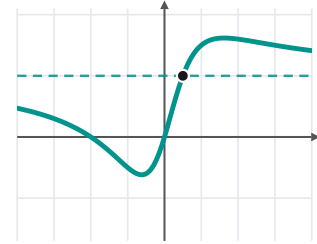
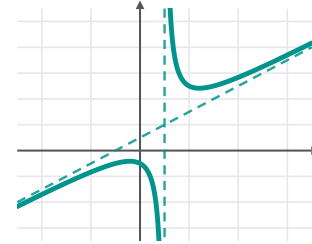
As x increases without bound, $g(x)$ decreases without bound:
 $\lim_{x \rightarrow \infty} g(x) = -\infty$.

Add a second limit only if it is right: an extra wrong statement can lose the point.

Exam trap

- Taking the first term written as the leading term.** In $5 + 3x^2 - 2x^3$ the leading term is $-2x^3$, the highest power, so the right end goes down.
- Forgetting to cube the coefficient.** $(2x - 1)^3$ starts with $(2x)^3 = 8x^3$, not $2x^3$.

1.7 Rational functions and end behavior

For large $|x|$, compare the leading terms of the top and the bottomtop degree smaller
 $(3x + 1)/(x^2 + 1)$  $3x/x^2 = 3/x \rightarrow 0$
asymptote $y = 0$ equal degrees
 $(2x^2 + 8x)/(x^2 + 4)$  $2x^2/x^2 = 2$
asymptote $y = 2$, crossed at $x = 1$ top degree one bigger
 $(x^2 + 1)/(x - 1)$  $x^2/x = x$: a slant line
dividing gives $y = x + 1$

Know by heart

If the degree of the top is...

less than the bottom's
 $y = 0$ equal to it
 $y = \text{the ratio of the leading coefficients}$
a slant line: divide

one more

1.11

at least two more
no line; it acts like the quotientFor a horizontal asymptote $y = b$, write
 $\lim_{x \rightarrow \pm\infty} r(x) = b$.

Worked example

Up a 1-mile hill at 15 mph, down at v mph. Average speed: $A(v) = \frac{30v}{v + 15}$. Can you average 30 mph?

- Top and bottom both have degree 1, so A acts like $30v/v = 30$.
- $A(45) = 22.5$, $A(90) \approx 25.7$, $A(1000) \approx 29.56$.

No. $A(v)$ gets closer and closer to 30, but $A(v) = 30$ means $30v = 30v + 450$, which is impossible.

Rational end behavior in limits

$$r(x) = \frac{x^3 + 2}{x + 5} \text{ acts like } \frac{x^3}{x} = x^2:$$

$$\lim_{x \rightarrow -\infty} r(x) = \infty \text{ and } \lim_{x \rightarrow \infty} r(x) = \infty$$

There is no asymptote line, because x^2 is neither a constant nor a line. By contrast, $(3 - 2x^2)/(x^2 + 1)$ tends to -2 at both ends. Free-response question 1 can ask for exactly this.

Exam trap

- Thinking a graph can't cross its horizontal asymptote. It can. $(2x^2 + 8x)/(x^2 + 4) = 2$ at $x = 1$, yet it still tends to 2 at both ends.
- Using $x^2/x = x$ as the slant asymptote. It gives only the slope. Divide: $\frac{x^2+1}{x-1} = x + 1 + \frac{2}{x-1}$, so the line is $y = x + 1$.
- Missing a minus sign. In $3x/(1 - x)$ the bottom leads with $-x$, so the graph tends to -3 , not 3.

1.8 Rational functions and zeros

Zeros come from the top, but only where the bottom is not zero

Know by heart

Real zeros of $r = P/Q$: the real zeros of P where $Q \neq 0$. A zero of both top and bottom is never a zero of r : it is a hole or an asymptote 1.9 1.10

Exam trap

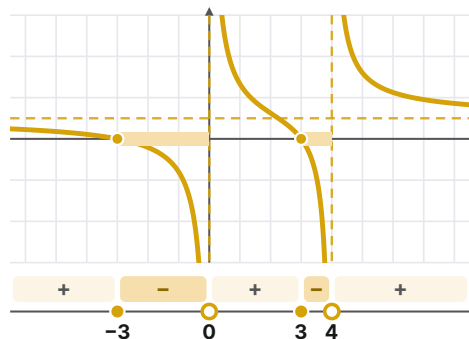
Forgetting to cut out a hole. $(x - 2)/(x - 2)(x - 3) \leq 0$ on $[-1, 2) \cup (2, 3]$, not $[-1, 3]$: the function does not exist at $x = 2$, even though the factor cancels.

Check the domain before you call it a zero

$$\frac{x^2 - 4}{x^2 + 3x + 2} = \frac{(x - 2)(x + 2)}{(x + 1)(x + 2)}$$

The only zero is $x = 2$. The top is also 0 at -2 , but -2 is not in the domain.Solve $r(x) \leq 0$ or $r(x) \geq 0$

- Get 0 on one side and one factored fraction on the other.
- Mark top zeros (filled for $\leq, \geq$) and bottom zeros (always open).
- Test one input in each interval.
- Write the union of the intervals that work.

filled dot: top is 0 (included); open: bottom is 0
 $(x^2 - 9)/(x^2 - 4x) \leq 0$ on $[-3, 0) \cup [3, 4)$, where the graph is on or below the axis.

Exam trap

Assuming the sign flips at every zero or asymptote. A squared factor like $(x - 2)^2$ is positive on both sides of 2, so only odd powers change the sign. For $r(x) = (x + 1)/(x - 2)^2$, $r(1.9) = 2.9/0.01 = 290$ and $r(2.1) = 310$: positive on both sides, so r changes sign only at $x = -1$.

Worked example

Solve $\frac{x + 4}{x - 1} \leq 3$.

- Subtract 3 and combine:
 $\frac{x + 4 - 3(x - 1)}{x - 1} = \frac{7 - 2x}{x - 1} \leq 0$
- Boundaries: 3.5 is included (the fraction is 0 there); 1 is not (the bottom is 0).
- Test 0: $7/(-1) < 0$ ✓. Test 2: $3/1 > 0$ ✗. Test 4: $-1/3 < 0$ ✓.

$$(-\infty, 1) \cup [3.5, \infty)$$

Exam trap

Multiplying both sides by $x - 1$. For $x < 1$ that multiplies by a negative number, which flips $\leq$; keeping $\leq$ loses $(-\infty, 1)$. Use a sign chart instead.

Worked example

Back on the hill, how fast must you ride down to average 25 mph?

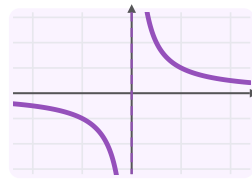
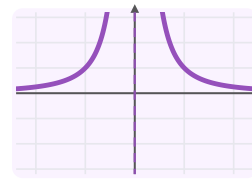
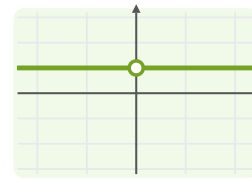
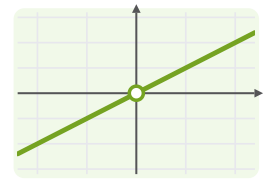
$$1 \quad \frac{30v}{v + 15} = 25, \text{ so } 30v = 25v + 375$$

(fine to multiply here: $v + 15 > 0$)

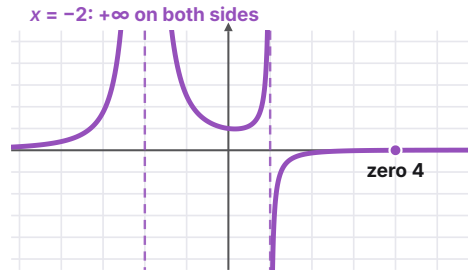
$$v = 75 \text{ mph}$$

1.9 Rational functions and vertical asymptotes

Where the bottom is 0 after cancelling, the output grows without bound

 $x = a$ makes the bottom zero.Count the power of $(x - a)$: m on top, n on the bottom. Then compare. **$n > m$: vertical asymptote $x = a$**  $1/(x - a)$: odd power
sides go opposite ways $1/(x - a)^2$: even power
sides go the same way **$n \leq m$: hole at (a, L)**  $(x - a)/(x - a)$: cancels
 L = simplified value at a  $(x - a)^2/(x - a)$
hole on the axis, not a zero**Know by heart** $x = a$ is a **vertical asymptote** when a is a zero of the bottom and either not a zero of the top, or a zero with a higher power on the bottom. $\lim_{x \rightarrow a^-} r(x)$ uses inputs just **below** a , $\lim_{x \rightarrow a^+} r(x)$ inputs just **above**. At an asymptote each is ∞ or $-\infty$.**Read each side**

- 1 Cancel common factors; keep the power of $(x - a)$ left on the bottom.
- 2 Put $x = a$ into everything else; the result is either positive or negative.
- 3 $(x - a)$ is positive just right of a and negative just left; an even power is positive on both sides.
- 4 Multiply the signs: $+$ means ∞ , $-$ means $-\infty$.

 $x = -2$: $+\infty$ on both sides $x = 1$: $+\infty$ on the left, $-\infty$ on the right**Worked example**One-sided limits of $r(x) = \frac{x - 4}{(x - 1)(x + 2)^2}$ at $x = 1$.

- 1 Everything else at $x = 1$: $(x - 4)/(x + 2)^2 = -3/9$, which is negative.
- 2 $(x - 1)$ is positive just right of 1, negative just left.

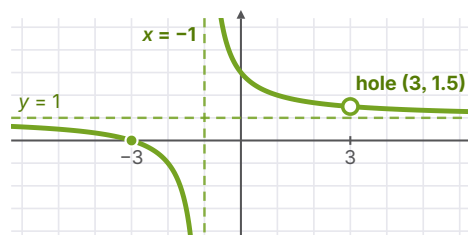
$$\lim_{x \rightarrow 1^-} r(x) = -\infty, \lim_{x \rightarrow 1^+} r(x) = \infty$$

A model with an asymptoteA 1960 model of world population, written as a rational function: $N(t) = 180/(2026.87 - t)$ billion people in year t . Its authors called the paper "Doomsday: Friday, 13 November, A.D. 2026".It fit the data before 1960, but 10, 1 and 0.1 years before 2026.87 it gives 18, 180 and 1,800 billion: $\lim_{t \rightarrow 2026.87^-} N(t) = \infty$. Past the asymptote it gives negative numbers: the asymptote marks where the model breaks down.**Exam trap**

- **Writing $r(a) = \infty$.** $r(a)$ doesn't exist. Write a one-sided limit for each side instead, such as $\lim_{x \rightarrow a^+} r(x) = \infty$.
- **Calling every shared factor a hole.** $(x - 2)/(x - 2)^2$ simplifies to $1/(x - 2)$, which still has $(x - 2)$ on the bottom: an asymptote.

1.10 Rational functions and holes

A factor that cancels completely leaves one missing point

Know by heartThere is a **hole** at a zero c of the bottom when $(x - c)$ has at least as high a power on top. The point (c, L) is missing: $r(c)$ does not exist, but L , the simplified expression at c , is $\lim_{x \rightarrow c} r(x)$.**Why cancelling doesn't fill the hole**Let $a = b$. Then $a^2 - b^2 = ab - b^2$, so $(a + b)(a - b) = b(a - b)$. Cancel $a - b$: $a + b = b$, so $2b = b$ and $2 = 1$.Cancelling $a - b$ divided by 0, because $a = b$. A factor can be cancelled only where it isn't 0, so that input stays out of the domain.**Zero, asymptote or hole?**Put $x = a$ into the top and the bottom. If only the top is 0, it is a **zero**; if only the bottom, a **vertical asymptote**; if both, cancel and compare the powers.**Worked example**Locate the hole of $r(x) = (x^2 - 9)/(x^2 - 2x - 3)$.

- 1 $(x - 3)(x + 3)/(x - 3)(x + 1)$: $(x - 3)$ cancels fully.
- 2 $(x + 3)/(x + 1)$ at 3 is $6/4 = 1.5$ (a table: 1.5013 at 2.99, 1.4988 at 3.01).

Hole at $(3, 1.5)$; asymptote $x = -1$; zero -3 **Exam trap**

- **Thinking cancelling fills the hole.** The worked example's $r(x)$ simplifies to $(x + 3)/(x + 1)$, but $r(3)$ is still undefined.
- **Calling a hole on the x-axis a zero.** $(x - 2)^2/((x + 6)(x - 2))$ has a hole at $(2, 0)$: $r(2)$ doesn't exist, so 2 is not a zero.
- **Copying the calculator's graph.** It draws straight through a hole; draw the open circle yourself.

Quick checkAt $x = 2$, hole or asymptote?

(a) $\frac{x - 2}{x^2 - 4}$ (b) $\frac{(x - 2)^2}{x^2 - 4}$ (c) $\frac{x - 2}{(x - 2)^2}$

Answer: (a) hole $(2, \frac{1}{4})$; (b) hole $(2, 0)$; (c) vertical asymptote.

1.11 Equivalent representations of polynomial and rational expressions

Standard form degree, ends, y-intercept

$$f(x) = \frac{x^3 - 2x^2 - 5x + 6}{x^2 - 1}$$

Degree 3 over degree 2, so there is a slant asymptote. The y-intercept is $f(0) = 6/(-1) = -6$.

Factored form zeros, holes, asymptotes, domain, sign

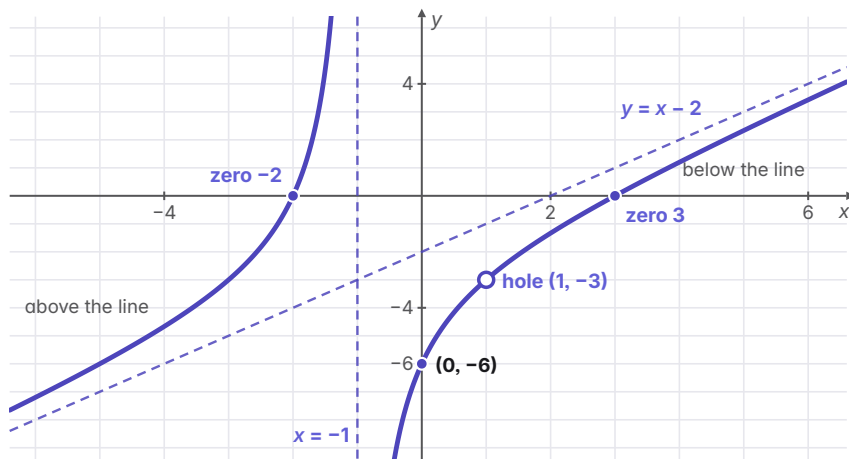
$$f(x) = \frac{(x-1)(x+2)(x-3)}{(x-1)(x+1)}$$

The zeros are -2 and 3. $(x-1)$ cancels, so there is a hole at $x = 1$ with height $\frac{(3)(-2)}{2} = -3$. The vertical asymptote is $x = -1$; the domain is every x except ± 1 .

Quotient + remainder slant asymptote

$$f(x) = x - 2 - \frac{4}{x+1} \quad (x \neq -1)$$

The slant asymptote is $y = x - 2$. The leftover fraction $-\frac{4}{x+1}$ is negative for $x > -1$, so the graph is below the line there, and positive for $x < -1$, so it is above.



Which form answers which question

Question	Form
zeros, x-intercepts, sign	factored
holes, vertical asymptotes, domain	factored, top and bottom
degree, leading term, end behavior	standard
y-intercept	standard, or plug in 0
slant asymptote and which side	quotient + remainder

Exam trap

- Dropping the restriction.** $(x^2 - 1)/(x - 1) = x + 1$ only for $x \neq 1$; write that next to it.
- Subtracting only the first term.** Subtract the whole row: $(x^3 - 2x^2 - 5x)$ minus $(x^3 + 0x^2 - x)$ is $-2x^2 - 4x$, not $-2x^2 - 6x$.
- Skipping a missing power.** Write $x^2 - 1$ as $x^2 + 0x - 1$ so each term lines up under its power.

Worked example

Making x items costs $C(x) = 0.01x^2 + 5x + 200$ dollars. What happens to the average cost?

- $A(x) = \frac{C(x)}{x} = 0.01x + 5 + \frac{200}{x}$.
- The fixed \$200 is shared over more items: $200/x$ shrinks to 0.

For large x , A follows $y = 0.01x + 5$

$$\begin{array}{r} x \quad -2 \\ x^2 + 0x - 1 \overline{) x^3 - 2x^2 - 5x + 6} \\ \underline{-(x^3 + 0x^2 - x)} \\ -2x^2 - 4x \\ \underline{-(-2x^2 + 0x + 2)} \\ -4x + 4 \\ \underline{-(-4x + 4)} \\ 0 \end{array}$$

Placeholder
0x keeps the columns lined up.

Know by heart

Dividing p by d gives a quotient q and a remainder r :

$$p = d \cdot q + r, \quad r = 0 \text{ or } \deg r < \deg d$$

so $\frac{p}{d} = q + \frac{r}{d}$. Here:

$$x^3 - 2x^2 - 5x + 6 = (x^2 - 1)(x - 2) + (-4x + 4),$$

$$\text{and } \frac{-4x + 4}{x^2 - 1} = -\frac{4}{x+1} \text{ for } x \neq 1.$$

Long division, step by step

- Write both polynomials in descending powers, with 0 for any missing power.
- Divide the leading terms to get the next term of the quotient.
- Multiply the whole divisor by that term and subtract the whole row.
- Repeat until the remainder's degree is below the divisor's.

Quick check

Slant asymptote of $\frac{2x^2 + 3x - 1}{x - 1}$?

Answer: $2x^2 + 3x - 1 = (x - 1)(2x + 5) + 4$, so $y = 2x + 5$.

$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1
$n = 6$	1 6 15 20 15 6 1

Each entry is the sum of the two above it. Row n lists the coefficients of $(a + b)^n$; entry k (counting from 0) is written $\binom{n}{k}$.

Know by heart

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Worked example

Expand $(x - 2)^4$.

- Row 4: 1, 4, 6, 4, 1. Powers of -2: 1, -2, 4, -8, 16. Multiply them in order: 1, -8, 24, -32, 16 are the coefficients.
- Check with $x = 1$: $(1 - 2)^4 = 1$ and $1 - 8 + 24 - 32 + 16 = 1$ ✓

$$x^4 - 8x^3 + 24x^2 - 32x + 16$$

One term, without expanding

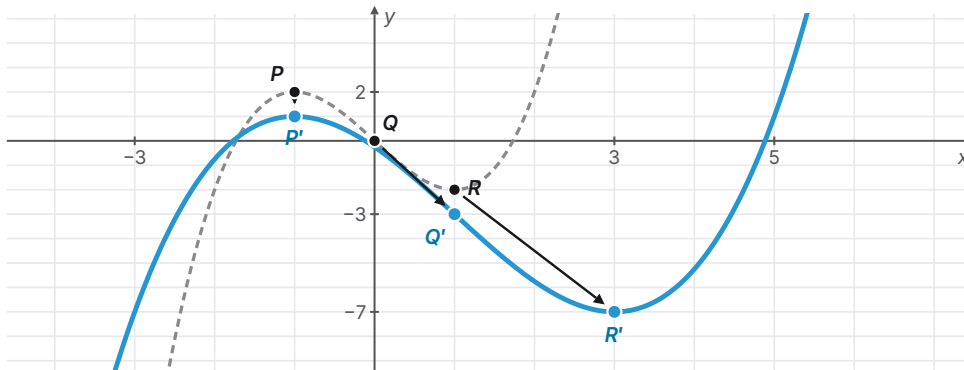
In $(x + 3)^5$ the powers of x and 3 add to 5, so the x^2 term has 3^3 :
 $\binom{5}{3} x^2 \cdot 3^3 = 10 \cdot 27x^2 = 270x^2$.

Exam trap

Cubing only the x , or dropping the middle terms.

$(2x + 1)^3 = 8x^3 + 12x^2 + 6x + 1$, not $2x^3 + 1$; $(x + 2)^4$ is not $x^4 + 16$. Use row 3 of Pascal's triangle for a cube.

1.12 Transformations of functions

 $g(x) = Af(B(x-h)) + k$ moves every point by the same rule

--- $f(x) = x^3 - 3x$

— $g(x) = 2f(\frac{1}{2}(x-1)) - 3$

$P(-1, 2) \rightarrow P'(-1, 1); Q(0, 0) \rightarrow Q'(1, -3); R(1, -2) \rightarrow R'(3, -7)$

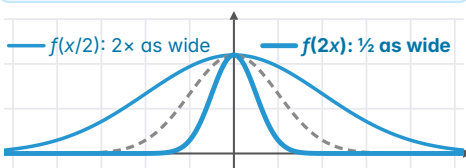
Move a few key points, then redraw the curve through them. Here $A = 2$, $B = \frac{1}{2}$, $h = 1$, $k = -3$, so each point (u, v) of f lands at $(1 + 2u, 2v - 3)$ on g : stretched to twice the width and twice the height, then moved right 1 and down 3.

One change at a time

$g(x)$	Graph of f is	$(u, v) \rightarrow$
$f(x) + k$	moved up k	$(u, v + k)$
$f(x - h)$	moved right h	$(u + h, v)$
$f(x + 3)$	moved left 3	$(u - 3, v)$
$Af(x)$	stretched vertically by $ A $; flipped if $A < 0$	(u, Av)
$f(Bx)$	stretched sideways by $1/ B $; flipped if $B < 0$	$(u/B, v)$
$-f(x)$	flipped top to bottom	$(u, -v)$
$f(-x)$	flipped left to right	$(-u, v)$

Domain and range move too

f : domain $[0, 6]$, range $[-1, 4]$. Then $g(x) = 2f(x/3) + 1$ has domain $[0, 18]$ (inputs $\times 3$) and range $[-1, 9]$ (outputs $\times 2, +1$).

Dashed: f .Vertex form is a transformed x^2

$a(x-h)^2 + k$: x^2 stretched by a , vertex moved to (h, k) . Complete the square: take out the 2, then add and subtract 9 inside, which subtracts 18 overall: $2x^2 - 12x + 11 = 2(x^2 - 6x + 9) - 18 + 11 = 2(x-3)^2 - 7$, so the vertex is $(3, -7)$.

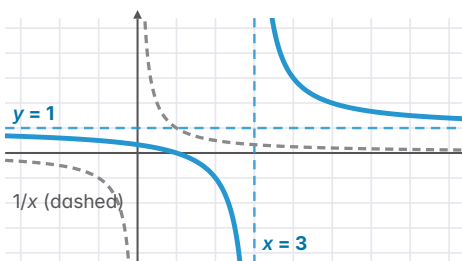
Worked example

$f(x) = x^2$ and $g(x) = -2f(3(x-1)) + 4$. Where does $(1, 1)$ go? Vertex, range, zeros?

1 $(1 + \frac{1}{3}, -2 \cdot 1 + 4) = (\frac{4}{3}, 2)$

2 $g(x) = -18(x-1)^2 + 4$: vertex $(1, 4)$, opens down.

3 $18(x-1)^2 = 4$, so $x = 1 \pm \frac{\sqrt{2}}{3}$.

Range $y \leq 4$; zeros ≈ 0.529 and 1.471 

Asymptotes move with the graph. $g(x) = 2/(x-3) + 1$ is $1/x$ stretched by 2, moved right 3 and up 1, so its asymptotes are $x = 3$ and $y = 1$. As one fraction it is $(x-1)/(x-3)$.

Worked example

A parabola has vertex $(2, 5)$ and passes through $(4, -3)$. Find it.

1 Start from the vertex:

$y = a(x-2)^2 + 5$

2 $-3 = a(4-2)^2 + 5$, so $4a = -8$.

$y = -2(x-2)^2 + 5$

Quick check

Describe $g(x) = -(x+2)^3 + 1$ as transformations of x^3 .

Answer: reflect over the x -axis, then shift left 2 and up 1; the point $(0, 0)$ goes to $(-2, 1)$.

Where each point goes

For $B \neq 0$:

$(u, v) \rightarrow (h + \frac{u}{B}, Av + k)$

- A and k , **outside** f , change the outputs just as written: multiply by A , then add k .
- B and h , **inside**, change the inputs the opposite way: divide by B , then add h .
- $A < 0$ reflects over the x -axis; $B < 0$ over the y -axis.

Speed changes are horizontal dilations

If $f(t)$ is a 3:00 song's sound at time t , double speed plays $f(2t)$ and lasts 1:30. Its graph is squeezed to half the width.

Exam trap

- Shifting the wrong way.** $f(x+3)$ moves the graph 3 to the **left**: now $x = -3$ gives $f(0)$.
- Shifting by the number you see.** $f(2x-6) = f(2(x-3))$ is a shift right 3, not 6. Factor first.
- Guessing where a point lands.** Under $f(2x-6)$ the point $(3, 6)$ of f goes to the x where $2x-6 = 3$: $(4.5, 6)$, not $(7.5, 6)$.
- Stretching instead of squeezing.** $f(3x)$ is $\frac{1}{3}$ as wide as f : x only has to reach a third as far to give f the same input.

Say it in the exam's words

For $g(x) = f(x/2) - 3$:

The graph of g is a **horizontal dilation** of the graph of f by a factor of 2 and a **vertical translation** down 3 units.

Translation means a shift, dilation a stretch or squeeze, reflection a flip. The same transformations build the sine models of Unit 3.

Quick check

$g(x) = 3f(x/2) - 4$ and $f(4) = 5$. Which value of g can you find, and what is it?

Answer: $x/2 = 4$ at $x = 8$:

$g(8) = 3 \cdot 5 - 4 = 11$.

Reflections and symmetry

Even: $f(-x) = f(x)$, so a flip over the y -axis changes nothing. Odd: $f(-x) = -f(x)$, so flips over both axes change nothing.

1.13

Function model selection and assumption articulation

The data suggest the type of model; the context sets its limits

Which model to try

roughly constant rate	linear
rates changing roughly linearly, or symmetric with one max or min	quadratic
area, two dimensions	quadratic
volume, three dimensions	cubic
several zeros or turns	higher degree
equal input steps; constant nonzero n th differences	degree n
different behavior on different intervals	piecewise

⚠ Exam trap

Trusting a perfect fit. Any $n + 1$ points with different inputs fit a polynomial of degree n or less exactly, yet it can be absurd one step outside the data. Choose the type from the pattern and the context first.

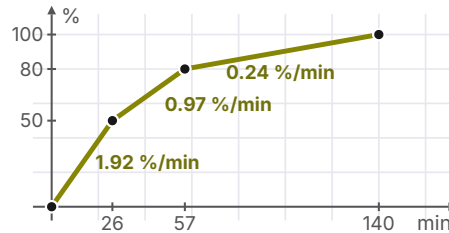
Pizza cuts
quadratic

1	2	4	7	11	16
1	2	3	4	5	
1	1	1	1		

The most pieces from n straight cuts. The pizza numbers have constant second differences (quadratic); the cake numbers have constant third differences (cubic).

Cake cuts
cubic

1	2	4	8	15	26
1	2	4	7	11	
1	2	3	4		
1	1	1			



A phone charges fast to 50%, slower to 80%, then slowly to 100%. Three rates on three intervals call for a piecewise model.

Assumptions and restrictions

- Assumptions:** what the model holds constant (a rate, a price) and how the quantities change together.
- Domain:** from the context (time ≥ 0), from the model itself (where it turns back), or from the range of the data.
- Range:** counts must be whole numbers. 1,128 students at 36 per bus is 31.33, so 32 buses.

✍ Explain a domain restriction

A ball thrown up from the ground has height $h(t) = -16t^2 + 48t$ feet:

Height can't be negative, but $h(t) < 0$ after $t = 3$, so the model applies only for $0 \leq t \leq 3$.

Name the context fact and what the model does there. "Because of the vertex" alone earns nothing.

1.14 Function model construction and application

Building a model, then using it

Four ways to build a model

- From conditions:** one equation for each data point, then solve the system.
- Transform a parent function** 1.12, as in vertex form or k/x .
- Regression** on the calculator: linear, quadratic, cubic or quartic (keys on page 10).
- Piecewise:** one rule per interval.

Write a piecewise model

The phone above, with t in minutes:

$$P(t) = \frac{50}{26}t \text{ for } 0 \leq t \leq 26$$

$$P(t) = 50 + \frac{30}{31}(t - 26) \text{ for } 26 < t \leq 57$$

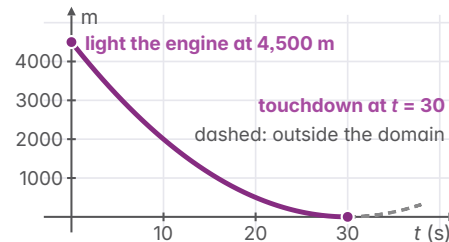
$$P(t) = 80 + \frac{20}{83}(t - 57) \text{ for } 57 < t \leq 140$$

Rational models

Inversely proportional: $y = k/x$. Inverse square: $y = k/x^2$ (gravity, light, sound). One data point gives k .

⚠ Exam trap

- Rounding the coefficients.** In the plays model on page 10, rounding $b = 2.75$ to 2.8 turns $D(10) = 40$ into 40.5. Keep every digit stored.
- Picking the highest r^2 .** A quartic always fits five points at least as well as a quadratic, but its extra turns may make no sense in the context.



Worked example

A booster starts 4,500 m high, falling at 300 m/s, and slows by 10 m/s each second (simplified). Model its height to touchdown.

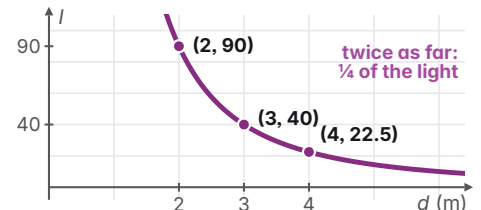
- It stops after $300/10 = 30$ s. Slowing steadily from 300 m/s to 0, it averages 150 m/s and falls $150 \cdot 30 = 4500$ m, so it stops as it lands.
- So the lowest point is $(30, 0)$:
 $h = a(t - 30)^2$, and
 $h(0) = 900a = 4500$.

$$h(t) = 5(t - 30)^2 \text{ for } 0 \leq t \leq 30$$

When is it at 1,000 m? $5(t - 30)^2 = 1000$ gives $t = 30 \pm \sqrt{200}$. Keep $t \approx 15.858$ s; the other root, 44.142, is after touchdown.

Using the model

The average rate is -250 m/s on $[0, 10]$ and -50 m/s on $[20, 30]$. The rates increase, so the graph is concave up: the fall slows.



Worked example

Light intensity is 90 units at 2 m and follows $I = k/d^2$. Intensity at 3 m?

- $90 = k/2^2$, so $k = 360$.
- $I(3) = 360/3^2$.

40 units at 3 m

Quick check

Gravity follows $F = k/d^2$. Move 3 times as far away: the force is multiplied by what?

Answer: $\frac{k}{(3d)^2} = \frac{1}{9} \cdot \frac{k}{d^2}$: one ninth.

Secant estimates and concavity

Estimate with the secant:

$y = f(a) + \text{rate} \cdot (x - a)$. Between the endpoints a **concave down** graph lies **above** its secant, so the estimate is too low; concave up, too high.

Page 10: the secant gives 28.375 at $t = 1.5$, below $D(1.5) \approx 28.844$.

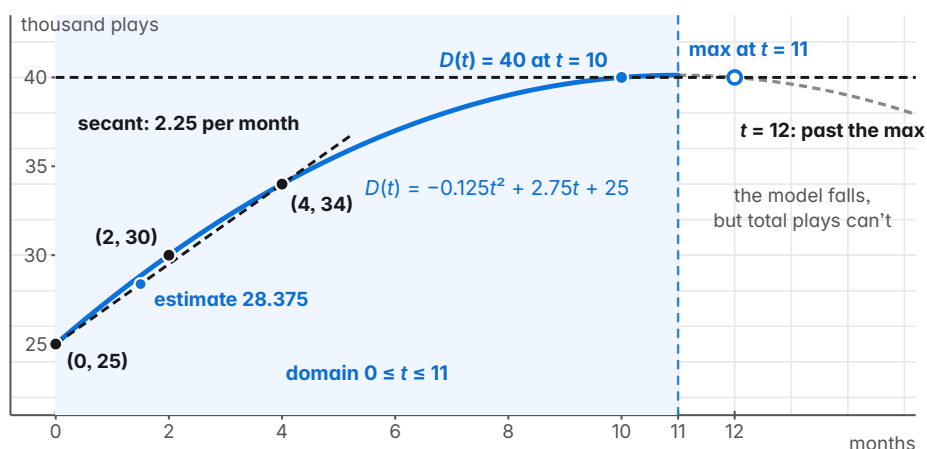
Free-response questions 1 and 2

Both are in the calculator part of the exam, and Unit 1 runs through both

Question 2, a model built from data, set out as the May 2027 exam asks it

Total plays of a song, in thousands, were 25, 30 and 34 at $t = 0, 2$ and 4 months. The model is $D(t) = at^2 + bt + c$. Each row below is one of the question's 6 points.

Part	An answer that earns the point	What earns it, and what loses it
A equations	$c = 25$, $4a + 2b + c = 30$, $16a + 4b + c = 34$	One equation built from each data point. A regression's output alone does not count as equations.
A values	$a = -0.125$, $b = 2.75$, $c = 25$	Any calculator method is fine here, including a quadratic regression on the three points, which gives a , b and c exactly. Give decimals.
B average rate	$\frac{D(4) - D(0)}{4 - 0} = \frac{34 - 25}{4} = 2.25$	The quotient must appear with the data values in it.
B meaning	From $t = 0$ to $t = 4$ months, total plays increased by 2.25 thousand per month on average .	Say the interval, the quantity, which way it changed, the units and "on average".
C reaching 40	$D(t) = 40$, so $t = 10$ months	Write the equation, then solve it with the intersect tool. It also gives $t = 12$; the next row shows why that answer is thrown out.
C domain	Total plays can never decrease, but D decreases after its maximum at $t = 11$. So the model only applies for $0 \leq t \leq 11$, and $t = 12$ is outside it.	Both halves: a fact about the context and what the model does. "Because of the vertex" alone earns nothing.



The model reaches 40 thousand plays at $t = 10$, and again at $t = 12$, after its maximum at $t = 11$; past the maximum the model no longer makes sense. The secant (slope 2.25) lies below the concave-down curve, so its estimate at $t = 1.5$ is too low.

Find a maximum or minimum

TI-84 Plus CE. Graph it and zoom out until no turn hides off the screen. Press **2nd** **trace** (CALC), choose 4: maximum or 3: minimum, press **enter** left of the turn, **enter** right of it, then **enter** again.

TI-Nspire CX. **menu** → Analyze Graph → Maximum (or Minimum), then click to the left of the turn and to the right of it.

Desmos. Tap the curve: its highest and lowest turning points show as grey dots. Tap one to read its coordinates.

Fit a quadratic to data

TI-84 Plus CE. **stat** 1: Edit. Enter the inputs in L1 and the outputs in L2. Then **stat** → CALC → 5: QuadReg, set Xlist L1 and Ylist L2, and choose Calculate.

TI-Nspire CX. On a Lists & Spreadsheet page, enter the data in columns named x and y . **menu** → Statistics → Stat Calculations → Quadratic Regression, with X List x and Y List y .

Desmos. Add a table ($\rightarrow$ table) with columns x_1 and y_1 , then type $y_1 \sim ax_1^2 + bx_1 + c$.

What question 1 asks from Unit 1

- Every zero of a given function, correct to three decimal places, and no wrong extra answers.
- End behavior in limit notation, for a polynomial or a rational function.
- Which model fits a table, and why: equal input steps with constant differences (or constant ratios, from Unit 2).
- Its other parts use composition and inverses from Unit 2.

Small things that cost points

- Decimal answers must be right to three places after the decimal point, rounded or cut off; fewer places lose the point.
- Rounding partway through pushes the final answer off. Store each value instead (**sto→** on a TI-84).
- A wrong extra answer next to a right one can cancel it.
- Write "=" only between equal things: $34 - 25 = 9 = 9/4$ says 9 equals $9/4$.

Find every zero

TI-84 Plus CE. Type the function in **y=** and press **graph**. Press **2nd** **trace** (CALC), 2: zero, then **enter** left of a crossing, right of it, and once more. Repeat for each crossing. Where the graph only touches the axis this fails (NO SIGN CHNG): use 3: minimum or 4: maximum.

TI-Nspire CX. On a Graphs page, type the function. **menu** → Analyze Graph → Zero, then click to the left of a crossing and to the right of it. Where the graph only touches the axis, use Minimum or Maximum instead.

Desmos. Type the function and tap the curve: every zero, including one where the graph only touches the axis, shows as a grey dot. Tap a dot to read it.

Solve an equation by graphing

TI-84 Plus CE. Put the left side in Y1 and the right side in Y2. Press **2nd** **trace** (CALC), choose 5: intersect, press **enter** on the first curve, **enter** on the second, move near the crossing and press **enter**.

TI-Nspire CX. Graph both sides as f_1 and f_2 . **menu** → Analyze Graph → Intersection, then click to the left of the crossing and to the right of it.

Desmos. Type each side as its own line, $y =$ left side and $y =$ right side. Tap where they cross to read the point.