

# Unit 2

AP Precalculus

## Exponential and Logarithmic Functions

Sequences · Exponential models · Composition · Inverses · Logarithms · Equations · Semi-log plots

### Unit 1

Polynomial and rational functions

30–40%

### Unit 2

Exponential and logarithmic functions

25–40%

### Unit 3

Trigonometric and polar functions

30–35%

### Unit 4

Parameters, vectors and matrices

not tested

Unit 2 is 25–40% of the multiple-choice section. Free-response questions 1, 2 and 4 all use it. Questions 1 and 2 allow a graphing calculator; question 4 does not, and asks for exact answers to exponential and log equations.

### 2.1

Sequences

### 2.2

Linear vs. exponential

### 2.3

Exponential functions

### 2.4

Exponent rules

### 2.5

Exponential models

### 2.6

Residuals

### 2.7

Composition

### 2.8

Inverse functions

### 2.9

Log expressions

### 2.10

Inverses of exponentials

### 2.11

Log functions

### 2.12

Log rules

### 2.13

Equations

### 2.14

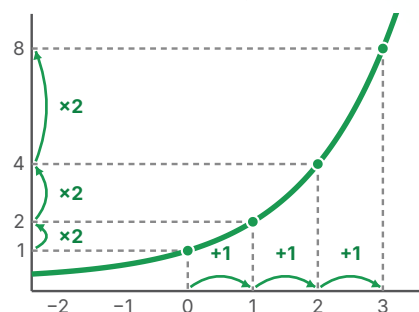
Log models

### 2.15

Semi-log plots

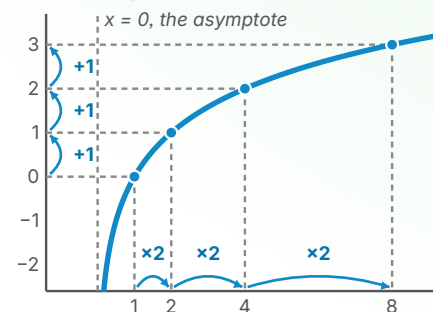
$$y = 2^x$$

Each step of 1 in  $x$  doubles  $y$ .



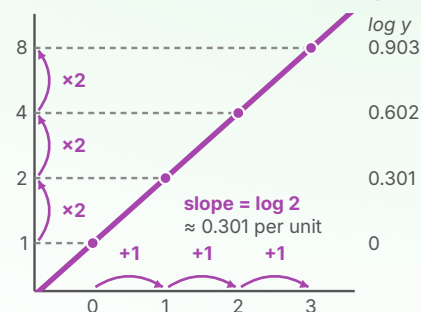
$$y = \log_2 x, \text{ the inverse of } 2^x$$

Each doubling of  $x$  adds 1 to  $y$ .



$$y = 2^x \text{ with a log scale on } y$$

Each step of 1 in  $x$  adds about 0.301 to  $\log y$ .



The first and third graphs plot  $(0, 1)$ ,  $(1, 2)$ ,  $(2, 4)$  and  $(3, 8)$ . The log graph plots the same pairs swapped, so it is the mirror image of  $2^x$  in the line  $y = x$ . The log scale spaces 1, 2, 4 and 8 evenly, so the exponential becomes a straight line with slope  $\log 2$ .

### What the exam asks most

**1 Solving exponential and log equations exactly.** Question 4 has no calculator. Get the exponential or log term by itself, undo it with a log or an exponential, and work out every number: write  $x = 9$ , not  $3^2$ . If your algebra gives  $e^x = -3$ , reject it and say why:  $e^x$  is positive for every  $x$ . [2.13](#)

**2 Building a model  $y = ab^t$  from two data points.** Write one equation for each point, divide them so  $a$  cancels, and give  $a$  and  $b$  to three decimal places. The value  $a$  belongs to  $t = 0$ , and the first row of a table is not always  $t = 0$ , so check before you use it. [2.5](#) [2.14](#)

**3 Rewriting as a single log or a single power.** Combine the terms with the exponent or log rules, then work out every number you can:  $\log_2 x + 3 \log_2 2$  should end as  $\log_2(8x)$ , not  $\log_2(2^3x)$ . [2.4](#) [2.12](#)

**4 Composition and inverses from tables, graphs and formulas.** For  $g(f(1))$ , find  $f(1)$  first and write it down: if  $f(1) = 3$  and  $g(x) = 2^x$ , then  $g(f(1)) = g(3) = 8$ . For an inverse value, look for it among the outputs and read the input:  $f(3) = -2$  means  $f^{-1}(-2) = 3$ . [2.7](#) [2.8](#)

**5 Explaining why a table is exponential.** Say that the inputs go up in equal steps and the outputs change by the same factor each step, and quote the ratios from the table. "The outputs decrease" is not a reason. [2.2](#) [2.5](#)

**6 Writing end behavior and asymptotes as limits.** Write all of it:  $\lim_{x \rightarrow -\infty} 2^x = 0$ . Near a log's asymptote, come from the side where the log exists:  $\ln(x - 1)$  needs  $x > 1$ , so write  $\lim_{x \rightarrow 1^+} \ln(x - 1) = -\infty$ , where  $1^+$  means from above 1. [2.3](#) [2.11](#)

### Formulas to know by heart

The exam gives you no formula sheet.

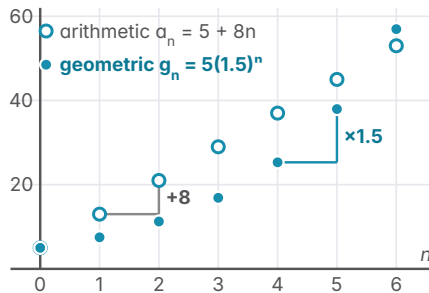
- **A log is an exponent:**  $\log_b c = a$  means  $b^a = c$  ( $b > 0$ ,  $b \neq 1$ ,  $c > 0$ ).
- **Exponents:**  $b^m b^n = b^{m+n}$ ,  $(b^m)^n = b^{mn}$ ,  $b^{-n} = 1/b^n$ .
- **Logs:**  $\log_b(MN) = \log_b M + \log_b N$ ,  $\log_b \frac{M}{N} = \log_b M - \log_b N$ ,  $\log_b M^p = p \log_b M$ .
- **Change of base:**  $\log_b x = \frac{\ln x}{\ln b}$ .
- **Growth factor:**  $b = 1 + r$ , with the rate  $r$  as a decimal: 4% growth gives  $b = 1.04$ .
- **Half-life:** an amount  $Q_0$  that halves every  $H$  units is  $Q_0 \left(\frac{1}{2}\right)^{t/H}$  after  $t$  units.
- $\log x$  with no base means base 10, and  $\ln x$  means base  $e \approx 2.718$ .
- **Residual** = actual – predicted.

### Not on the exam

Series (sums of a sequence's terms) and log scales on the  $x$ -axis are not tested, and you won't have to say what  $a$  and  $b$  mean in a log model  $y = a + b \ln x$ .

## 2.1 Arithmetic and geometric sequences

A sequence is a function whose inputs are whole numbers  $n = 0, 1, 2, \dots$ , so its graph is separate dots, not a connected curve.



The geometric sequence's jumps keep growing, so it passes the arithmetic one between  $n = 5$  and  $n = 6$ .

**Arithmetic:** add  $d$       **Geometric:** multiply by  $r$

$$a_n = a_0 + dn \quad g_n = g_0 r^n$$

$$a_n = a_k + d(n - k) \quad g_n = g_k r^{n-k}$$

The second row starts from any known term  $a_k$  or  $g_k$  and counts the  $n - k$  steps from it.

### Worked example

A geometric sequence has  $g_3 = 40$  and  $g_6 = 320$ . Write  $g_n$ .

- Term 6 is three steps after term 3, so  $r^3 = \frac{320}{40} = 8$  and  $r = 2$ .
- $g_n = 40 \cdot 2^{n-3}$ . Going back three steps,  $g_0 = 40 \div 8 = 5$ .

$$g_n = 5 \cdot 2^n, \text{ the same sequence as } 40 \cdot 2^{n-3}$$

### Exam trap

Counting  $n$  steps from term 1 instead of  $n - 1$ . Term  $n$  is only  $n - 1$  steps after  $a_1$ , so  $a_n = a_1 + (n - 1)d$  and  $g_n = g_1 r^{n-1}$ . A cent doubled daily from day 1 is  $2^{29}$  cents (\$5,368,709.12) on day 30, not  $2^{30}$ .

### Quick check

An arithmetic sequence has  $a_2 = 11$  and  $a_7 = 26$ . Find  $a_0$  and  $a_{20}$ .

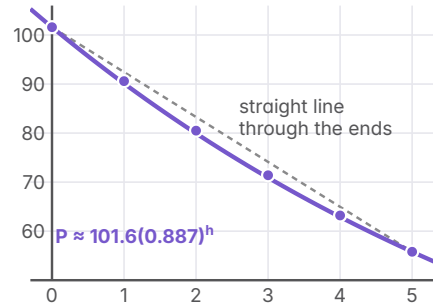
**Answer:**  $d = \frac{26-11}{7-2} = 3$ ,  
 $a_0 = a_2 - 2d = 11 - 6 = 5$ ,  $a_{20} = 5 + 60 = 65$ .

### A negative ratio

3, -6, 12, -24 is geometric with  $r = -2$ , but an exponential  $ab^x$  needs a positive base, so no exponential function passes through these terms.

## 2.2 Change in linear and exponential functions

If the input can be any real number, not just a whole number, an arithmetic sequence becomes a linear function, and a geometric sequence with a positive ratio (other than 1) becomes an exponential function.



Air pressure in kilopascals (kPa), measured by a weather balloon over Key West, Florida, on 18 February 2025, read at each whole kilometer and rounded. The points sag below the straight line joining the first and last points, so the data are concave up, as a decreasing exponential is.

| Height $h$ (km) | Pressure (kPa) | Difference | Ratio |
|-----------------|----------------|------------|-------|
| 0               | 101.6          |            |       |
| 1               | 90.6           | -11.0      | 0.892 |
| 2               | 80.5           | -10.1      | 0.889 |
| 3               | 71.4           | -9.1       | 0.887 |
| 4               | 63.2           | -8.2       | 0.885 |
| 5               | 55.8           | -7.4       | 0.883 |

The differences shrink, but the ratios stay near 0.89. From the first and last rows,  $b = (55.8/101.6)^{1/5} \approx 0.887$ , so  $P \approx 101.6(0.887)^h$ : each kilometer up, the pressure is about 88.7% of what it was a kilometer lower.

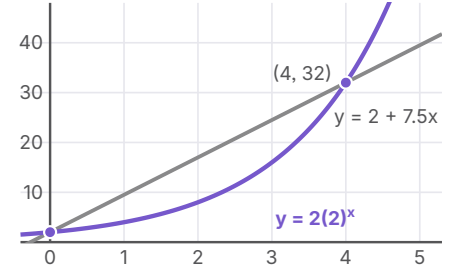
### Explaining why the model is exponential

Over equal input steps of 1 km, the outputs change by a **nearly constant factor** of about 0.89 ( $\frac{90.6}{101.6} \approx \frac{80.5}{90.6} \approx 0.89$ ), so an **exponential** model fits.

### Exam trap

- Giving "the outputs decrease" as the reason. Linear and exponential functions can both decrease. Name the equal input steps and the constant ratio, with numbers from the table, as in the sentence above.
- Treating a ratio over 2 units as  $b$ .  $b$  is the factor for one unit of input. If a table halves every 2 units,  $b^2 = 0.5$ , so  $b \approx 0.707$ , not 0.5.

Over equal input steps, a linear function adds the same amount and an exponential multiplies by the same factor



Exactly one line and one exponential pass through  $(0, 2)$  and  $(4, 32)$ . Past  $x = 4$  the exponential stays ahead.

### A line or an exponential through two points

**Line**  $y = y_1 + m(x - x_1)$ ,  
 $m = \frac{y_2 - y_1}{x_2 - x_1}$

**Exponential**  $y = y_1 b^{x-x_1}$   
 $b = (y_2/y_1)^{1/(x_2-x_1)}$

For the line you subtract the outputs; for the exponential you divide them (they must have the same sign). Over an input step of length  $L$ , an exponential multiplies by  $b^L$ .

### Worked example

An exponential function has  $f(2) = 12$  and  $f(5) = 96$ . Find  $f(x) = ab^x$ .

- Divide:  $\frac{ab^5}{ab^2} = b^3 = \frac{96}{12} = 8$ , so  $b = 2$ .
- Then  $a = \frac{12}{2^2} = 3$ .

$$f(x) = 3 \cdot 2^x, \text{ the same as } 12 \cdot 2^{x-2}$$

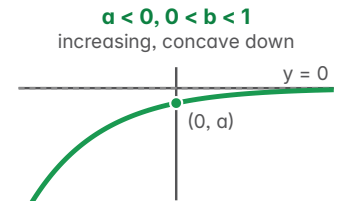
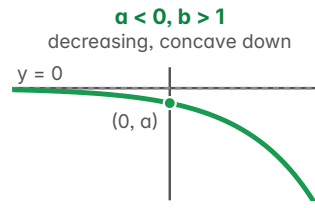
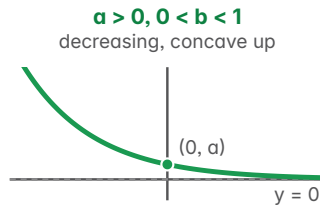
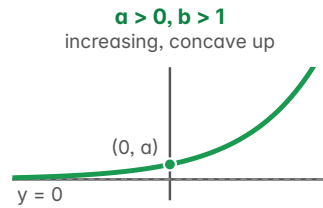
|                  | Linear             | Exponential                             |
|------------------|--------------------|-----------------------------------------|
| Form             | $f(x) = mx + c$    | $f(x) = ab^x$                           |
| Value at $x = 0$ | $c$                | $a$                                     |
| Each step of 1   | adds the slope $m$ | multiplies by $b$ ( $b > 0, b \neq 1$ ) |

### Quick check

The inputs 1, 3, 5, 7 give the outputs 4, 10, 25, 62.5. Linear or exponential? Find the factor per unit of input.

**Answer:** every 2-unit step multiplies the output by 2.5, so it is exponential with  $b = \sqrt{2.5} \approx 1.581$ .

## 2.3 Exponential functions

 $f(x) = ab^x$  with  $a \neq 0$ ,  $b > 0$  and  $b \neq 1$ :  $a$  is the initial value and  $b$  is the baseEvery  $f(x) = ab^x$  has

|                      |                                                                 |
|----------------------|-----------------------------------------------------------------|
| Domain               | all real numbers                                                |
| Range                | $y > 0$ if $a > 0$ ; $y < 0$ if $a < 0$                         |
| Intercept            | $(0, a)$ , and no $x$ -intercept                                |
| Asymptote            | horizontal, $y = 0$                                             |
| Shape                | always increasing or always decreasing; never changes concavity |
| Max, min, inflection | none                                                            |

## ⚠ Exam trap

**Thinking a decreasing exponential is concave down.**  $3(0.5)^x$  decreases, but its values 3, 1.5, 0.75 drop by less each time, so it is concave up. The concavity of  $ab^x$  depends only on the sign of  $a$ : up when  $a > 0$  and down when  $a < 0$ .

Limit of  $f(x)$ 

|                    | $x \rightarrow \infty$ | $x \rightarrow -\infty$ |
|--------------------|------------------------|-------------------------|
| $a > 0, b > 1$     | $\infty$               | 0                       |
| $a > 0, 0 < b < 1$ | 0                      | $\infty$                |
| $a < 0, b > 1$     | $-\infty$              | 0                       |
| $a < 0, 0 < b < 1$ | 0                      | $-\infty$               |

## ✍ Writing end behavior as a limit

$g(x) = 40(0.8)^x$  approaches 0 as  $x$  grows without bound, so write

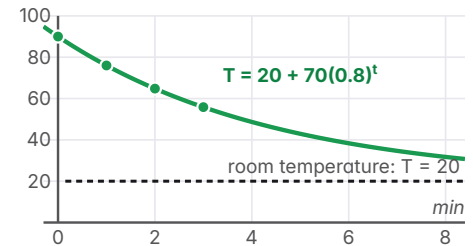
$$\lim_{x \rightarrow \infty} g(x) = 0$$

To earn the point, write all four parts: lim, the approach  $x \rightarrow \infty$ , the function's name and the value. Writing only " $g \rightarrow 0$ ", or a limit without  $g$  in it, loses the point.

## 🔍 Quick check

$f(x) = -2(3)^x + 5$ . Give the asymptote, the range and both end behaviors.

**Answer:** asymptote  $y = 5$ ; range  $y < 5$ ;  
 $\lim_{x \rightarrow -\infty} f(x) = 5$  and  $\lim_{x \rightarrow \infty} f(x) = -\infty$ .



Tea cooling in a 20 °C room. Its temperature above the room,  $T - 20$ , is multiplied by 0.8 each minute.

📌 An exponential shifted up by  $k$ 

$ab^x + k$  has the asymptote  $y = k$ . Its outputs no longer have a constant ratio, but  $f(x) - k$  does: for the tea,  $T - 20$  is 70, 56, 44.8, 35.84 at 0, 1, 2 and 3 minutes, each 0.8 times the last.

## ⚠ Exam trap

**Giving the range as  $y > 0$  after a shift.**  $ab^x + k$  levels off at  $y = k$ , so its range is  $y > k$  when  $a > 0$  (the tea:  $T > 20$ ) and  $y < k$  when  $a < 0$ .

## 2.4 Exponential function manipulation

The exponent rules work on products and quotients of powers, never on sums

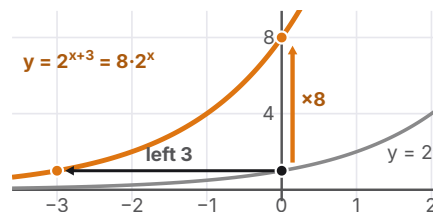
| Rule ( $b > 0$ )          | Example                              |
|---------------------------|--------------------------------------|
| $b^m \cdot b^n = b^{m+n}$ | $2^3 \cdot 2^4 = 2^7$                |
| $b^m / b^n = b^{m-n}$     | $5^7 / 5^4 = 5^3 = 125$              |
| $(b^m)^n = b^{mn}$        | $(3^2)^3 = 3^6$                      |
| $b^{-n} = 1/b^n$          | $2^{-3} = \frac{1}{8}$               |
| $b^{1/k} = \sqrt[k]{b}$   | $8^{1/3} = 2$ , $27^{2/3} = 3^2 = 9$ |
| $b^0 = 1$                 | $7^0 = 1$                            |

## 📌 Powers worth recognizing

$4 = 2^2$ ,  $8 = 2^3$ ,  $16 = 2^4$ ,  $32 = 2^5$ ,  $9 = 3^2$ ,  
 $27 = 3^3$ ,  $81 = 3^4$ ,  $25 = 5^2$ ,  $125 = 5^3$ ,  
 $\sqrt{3} = 3^{1/2}$ ,  $\frac{1}{9} = 3^{-2}$ ,  $\frac{1}{e^6} = e^{-6}$

## 📌 Changing the base or the time step

$b^{cx} = (b^c)^x$ , so  $2^{3x} = 8^x$ . To see the factor over  $h$  units, write  $ab^t$  as  $a(b^h)^{t/h}$ . Something that doubles every day is multiplied by  $2^7 = 128$  each week, because  $2^d = (2^7)^{d/7} = 128^{d/7}$  with  $d$  in days.



Shifting  $y = 2^x$  three units left gives the same graph as stretching it vertically by a factor of 8, because  $2^{x+3} = 2^3 \cdot 2^x = 8 \cdot 2^x$ .

## 📌 Worked example

No calculator. Write  $\frac{9^{x-1} \cdot 3^{2x}}{27}$  as a single power of 3.

- Use base 3:  $9^{x-1} = (3^2)^{x-1} = 3^{2x-2}$  and  $27 = 3^3$ .
- Add the exponents on top and subtract the one below:  
 $(2x - 2) + 2x - 3 = 4x - 5$ .

$$3^{4x-5}$$

## ⚠ Exam trap

- Using an exponent rule on a sum.  $2^x + 2^x = 2 \cdot 2^x = 2^{x+1}$ , not  $2^{2x}$ : at  $x = 3$ ,  $8 + 8 = 16$ , but  $2^6 = 64$ .
- Multiplying the coefficient into the base.  $3 \cdot 2^x$  is not  $6^x$ : at  $x = 2$ , 12 is not 36.
- Adding exponents for a power of a power.  $(2^x)^3 = 2^{3x}$ , not  $2^{x+3}$ : at  $x = 1$  they are 8 and 16.

## 📌 Worked example

A rough rule says air pressure halves every 5.5 km up:  $P(h) = 100 \left(\frac{1}{2}\right)^{h/5.5}$  percent of sea level. Find the drop per km and  $P$  on Everest (8.849 km).

$$1 \quad \left(\frac{1}{2}\right)^{h/5.5} = \left[\left(\frac{1}{2}\right)^{1/5.5}\right]^h \approx 0.8816^h$$

$$2 \quad P(8.849) = 100(0.5)^{8.849/5.5} \approx 32.8$$

Pressure drops about 11.8% per kilometer; the summit has 32.8% of sea-level pressure.

## 2.5 Exponential function context and data modeling

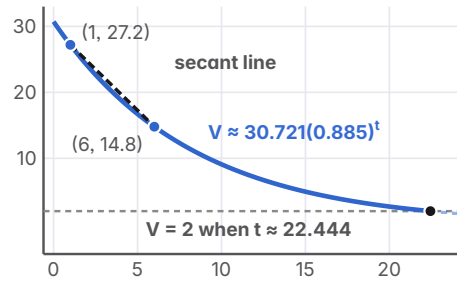
The base is the growth factor per unit of input:  $b = 1 + r$

### Five ways to build the model

- From a start and a percent rate.** An amount of 500 growing 4% a year is  $500(1.04)^t$ , since  $b = 1 + r$ ; losing 15% a year gives  $b = 0.85$ .
- From two data points.** Write one equation for each point, divide them so  $a$  cancels, solve for  $b$ , then find  $a$ .
- From a doubling time  $T$  or half-life  $H$ .** Use  $Q_0 2^{t/T}$  or  $Q_0 (\frac{1}{2})^{t/H}$ , where  $Q_0$  is the starting amount.
- From a level-off value.** If the data approach a value  $k$ , use  $ab^t + k$ .
- From data, with the calculator's exponential regression** (steps on page 10).

### The natural base $e$

$e \approx 2.718$ . Since  $ae^{kt} = a(e^k)^t$ , it is an ordinary exponential with base  $b = e^k$ , and  $k = \ln b$ . So  $e^{0.04t}$  grows by a factor of  $e^{0.04} \approx 1.0408$  per unit: about 4.08%, not 4%.



Value in thousands of dollars,  $t$  years after purchase. The model is concave up, so the secant line through the two data points lies above it.

### Exam trap

- Calling the first value in the table  $a$ .**  $a$  is the value at  $t = 0$ . The car's first value is at  $t = 1$ , so its equation is  $27.2 = ab$ .
- Rounding  $b$  before finding  $a$ .**  $27.2 \div 0.885 = 30.734$ , but  $a = 30.721$ : the second decimal is already wrong. Store  $b$  (on a TI-84, `sto→` `alpha` `apps` `enter` saves it as B) and divide by B.

### Worked example

A car is worth 27.2 thousand dollars 1 year after it is bought and 14.8 thousand dollars after 6 years. Find  $a$  and  $b$  in  $V(t) = ab^t$ .

- One equation per data point:  $ab = 27.2$  and  $ab^6 = 14.8$ .
- Divide:  $b^5 = \frac{14.8}{27.2}$ , so  $b = (\frac{14.8}{27.2})^{1/5} = 0.885398 \dots$
- $a = 27.2 \div b = 30.720645 \dots$

$a \approx 30.721$ ,  $b \approx 0.885$ : the car keeps 88.5% of its value each year.

### The rate and the domain in sentences

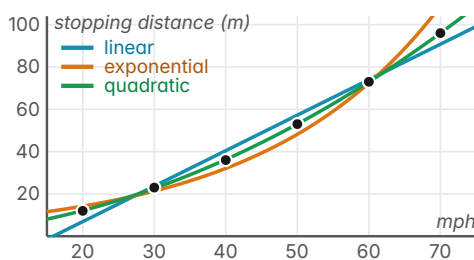
From  $t = 1$  to  $t = 6$  years, the car's value decreased by 2.48 thousand dollars per year **on average**.

The owner gives the car away when it is worth 2 thousand dollars.  $V(t) = 2$  gives  $t \approx 22.444$ , so the model applies for  $0 \leq t \leq 22.444$ .

## 2.6 Competing function model validation

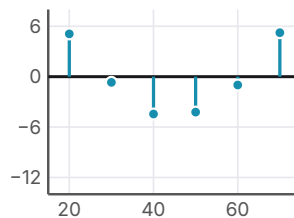
The right type of model leaves residuals with no pattern

### UK Highway Code stopping distances



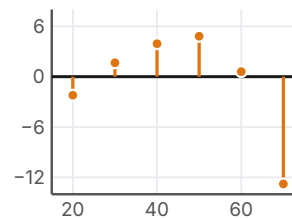
### Linear residuals

U-shaped, so wrong type



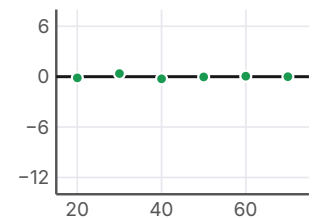
### Exponential residuals

arched, so wrong type



### Quadratic residuals

no pattern, so it fits



### Residuals

A **residual** is **actual – predicted**. Plot each residual against its input.

- Positive** the model **underestimates**
- Negative** the model **overestimates**
- No pattern** the type of model fits
- A curve** the model misses the shape

### Check each model against common sense

- A car at 0 mph stops in 0 m. The line predicts -26.6 m and the exponential 6.3 m, but the quadratic gives a sensible 0.6 m.
- Outside the data the models disagree badly: at 100 mph the line predicts 141 m, the exponential 369 m and the quadratic 184 m.

### Plot the residuals

**TI-84 Plus CE.** Run the regression you want to test with Xlist L1 and Ylist L2, such as `stat` → `CALC` → 5: QuadReg. Then `2nd` `y=` (STAT PLOT): turn Plot1 On with Xlist L1 and Ylist RESID (`2nd` `stat` → NAMES), and press `zoom` 9.

**TI-Nspire CX.** On a Data & Statistics page with  $x$  and  $y$  on the axes: `menu` → Analyze → Regression and pick the model, then `menu` → Analyze → Residuals → Show Residual Plot.

**Desmos.** After a fit such as  $y_1 \sim ax_1^2 + bx_1 + c$ , click "plot" next to the residuals  $e_1$ .

### Exam trap

**Trusting a high correlation  $r$ .** The linear fit here has  $r \approx 0.99$ , yet its residuals form a U, so a line is the wrong shape. Judge a model by its residual plot.

### Explaining why a model's error grows

Suppose a model matches sales at  $t = 91$  days, and then sales fall while the model rises.

The model and the actual values **agree** at  $t = 91$ . After that the model increases while the actual values decrease, so the gap between them, the error, **grows**.

### Exam trap

**Assuming "error" always means actual – predicted.** A question may define error as predicted – actual, or as the size of the residual, so use its definition and say which way the model is off. At 50 mph a car stops in 53 m and the line predicts 57.2 m, so the residual is -4.2 m but predicted – actual is 4.2 m.

## 2.7 Composition of functions

In  $f(g(x))$ ,  $g$  acts first and its output becomes  $f$ 's input



### Composition rules

- $(f \circ g)(x) = f(g(x))$ : work from the inside out.
- The domain of  $f(g(x))$  is every  $x$  that  $g$  accepts and that gives an output  $f$  accepts.
- Order matters. With the functions above,  $g(f(x)) = 3 \ln(x - 2) + 1$ , a different function.
- The identity  $I(x) = x$  changes nothing:  $f(I(x)) = I(f(x)) = f(x)$ .

### Exam trap

- Multiplying instead of composing.**  $f(g(x))$  feeds  $g$ 's output into  $f$ : above,  $f(g(1)) = f(4) = \ln 2$ , while  $f(1) \cdot g(1)$  doesn't exist ( $\ln(-1)$  is undefined).
- Taking the domain from  $g$  alone.** Above,  $g$  accepts every  $x$ , but  $f$  needs its input above 2, so  $f(g(x))$  needs  $3x + 1 > 2$ , which means  $x > 1/3$ .

| $x$    | 0 | 1 | 2 | 3  | 4 |
|--------|---|---|---|----|---|
| $f(x)$ | 5 | 3 | 0 | -2 | 1 |

### Worked example

$f$  is defined only at the inputs in the table, and  $g(x) = 2^x$ . Find  $g(f(1))$ ,  $f(g(1))$ ,  $f(f(4))$  and  $f(f(3))$ .

- $g(f(1)) = g(3) = 2^3 = 8$
- $f(g(1)) = f(2) = 0$ , not the same as  $g(f(1))$
- $f(f(4)) = f(1) = 3$
- $f(f(3)) = f(-2)$ , but  $-2$  is not an input in the table

8, 0, 3, and "not defined"

### Exam trap

**Writing only the final number.** A correct number with no work shown does not get the free-response point. Write the inside value too:  $g(f(1)) = g(3) = 8$ .

### Taking a function apart

$e^{x^2+1}$  outer  $e^u$ , inner  $x^2 + 1$   
 $\sqrt{\ln x}$  outer  $\sqrt{u}$ , inner  $\ln x$   
 $\log_2(5x)$  outer  $\log_2 u$ , inner  $5x$

Here  $u$  is the inside part. To put it back together, replace *every*  $u$  in the outer rule with the inner function.

### Transformations are compositions

A vertical shift  $f(x) + k$  is  $g(f(x))$  with  $g(x) = x + k$ , and a horizontal shift  $f(x + k)$  is  $g(f(x))$  with the same  $g$ . A vertical stretch  $c \cdot f(x)$  is  $g(f(x))$  with  $g(x) = cx$ .

### Quick check

$p(x) = x^2$  and  $q(x) = x - 3$ . Find  $p(q(x))$  and  $q(p(x))$ .

**Answer:**  $(x - 3)^2$  and  $x^2 - 3$ , which are different.

## 2.8 Inverse functions

If  $f(a) = b$ , then  $f^{-1}(b) = a$ : an inverse swaps inputs and outputs

### When a function has an inverse

A function has an inverse when each output comes from **exactly one** input. The domain of  $f^{-1}$  is the range of  $f$  (and the other way round), and  $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ .

In context, an inverse answers the reverse question: a taxi charging  $f(m) = 3 + 2m$  dollars for  $m$  miles gives  $f^{-1}(15) = 6$ : \$15 pays for 6 miles.

### Given To find the inverse

|         |                                                  |
|---------|--------------------------------------------------|
| Table   | swap the input and output rows                   |
| Graph   | reflect in $y = x$ : $(a, b) \rightarrow (b, a)$ |
| Formula | swap $x$ and $y$ , solve for $y$                 |

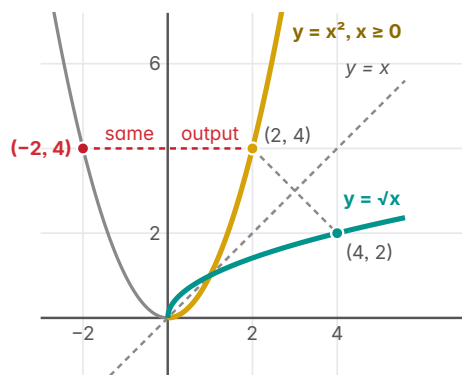
### Exam trap

- Reading  $f^{-1}$  as  $1/f$ .** The  $-1$  means inverse: from the table in 2.7,  $f^{-1}(3) = 1$  because  $f(1) = 3$ , not  $\frac{1}{3}$ .

### Quick check

Find  $f^{-1}(x)$  for  $f(x) = 4 - x^3$ .

**Answer:**  $x = 4 - y^3$ , so  $f^{-1}(x) = \sqrt[3]{4 - x}$ .



$x^2$  sends both  $-2$  and  $2$  to  $4$ , so on all reals it has no inverse. If we keep only  $x \geq 0$  (dropping the grey half), it has an inverse,  $\sqrt{x}$ .

### Undo the steps in reverse order

$$x \xrightarrow{\times 2} 2x \xrightarrow{+3} 2x + 3 \xrightarrow{f^{-1} \text{ undoes the last step first}} (x-3)/2 \xrightarrow{\div 2} x - 3 \xrightarrow{-3} x$$

For  $f(x) = 2x + 3$ , check by composing:  $f\left(\frac{x-3}{2}\right) = 2 \cdot \frac{x-3}{2} + 3 = x$ .

### Worked example

Find the inverse of  $f(x) = (x - 1)^2 + 3$  for  $x \geq 1$ .

- Swap  $x$  and  $y$ :  $x = (y - 1)^2 + 3$ , so  $(y - 1)^2 = x - 3$ .
- The inverse's outputs are  $f$ 's inputs, so  $y \geq 1$  and you take the positive root:  $y - 1 = \sqrt{x - 3}$ .
- Check:  $f^{-1}(7) = 3$  and  $f(3) = 7$ .

$$f^{-1}(x) = 1 + \sqrt{x - 3}, \text{ for } x \geq 3$$

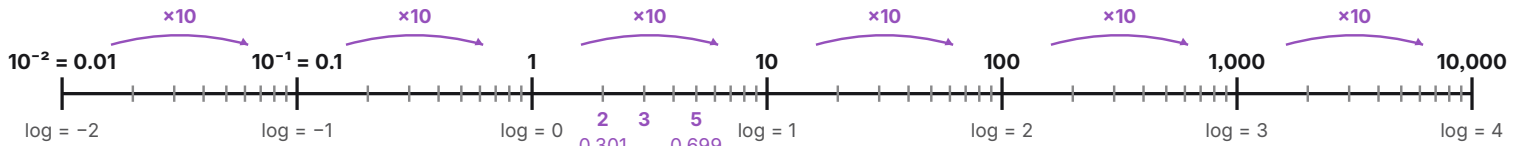
### Explaining whether $f$ has an inverse

**No:** for  $h(x) = x^2 + 3x + 1$ , the output 1 comes from two inputs,  $h(-3) = h(0) = 1$ , so  $h^{-1}(1)$  would need two values.

**Yes:**  $g(x) = 2^x + x$  is increasing on its whole domain (both  $2^x$  and  $x$  increase), so no two points on its graph share a  $y$ -value.

Writing only "one-to-one" or "passes the horizontal line test" does not get the point; name values or points of the function.

## 2.9 Logarithmic expressions

A logarithm is an exponent:  $\log_b c = a$  means  $b^a = c$ 

Each step to the right multiplies the number by 10 and adds 1 to its base-10 log. Inside each step, the small marks for 2, 3, ..., 9 crowd toward the right end: 2 already sits about a third of the way along ( $\log 2 \approx 0.301$ ), and 5 is past halfway ( $\log 5 \approx 0.699$ ).

$$\log_2 8 = 3 \Leftrightarrow 2^3 = 8$$

base raised to the log gives the argument

## The definition

$\log_b c = a$  means  $b^a = c$ , for  $b > 0$ ,  $b \neq 1$  and  $c > 0$ .

- $\log x$  means base 10, and  $\ln x$  means base  $e$ .
- $\log_b 1 = 0$  and  $\log_b b = 1$  for every base, because  $b^0 = 1$  and  $b^1 = b$ .

## Evaluate by hand

## because

|                                 |                        |
|---------------------------------|------------------------|
| $\log_2 32 = 5$                 | $2^5 = 32$             |
| $\log_3 \frac{1}{9} = -2$       | $3^{-2} = \frac{1}{9}$ |
| $\log 0.001 = -3$               | $10^{-3} = 0.001$      |
| $\log_5 \sqrt{5} = \frac{1}{2}$ | $5^{1/2} = \sqrt{5}$   |
| $\log_4 8 = \frac{3}{2}$        | $4^{3/2} = 2^3 = 8$    |
| $\ln \frac{1}{e} = -1$          | $e^{-1} = \frac{1}{e}$ |

To estimate a log, find the powers of 2 on either side of 20:  $16 < 20 < 32$ , so  $\log_2 20$  is between 4 and 5 (about 4.32).

## Exam trap

- Taking the log of 0 or a negative number.** No power of a positive base is 0 or negative, so  $\log 0$  and  $\log(-5)$  don't exist. Reject any solution that makes an argument 0 or negative:  $\log x + \log(x - 3) = 1$  gives  $x = 5$  or  $-2$ , and  $\log(-2)$  doesn't exist.
- Thinking a log can't be negative.**  $\log 0.01 = -2$  because  $10^{-2} = 0.01$ . With a base above 1, every argument between 0 and 1 has a negative log.

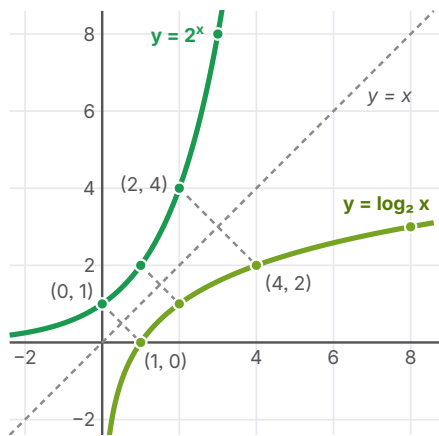
## Quick check

Evaluate  $\log_{16} 4$ ,  $\log_2 \frac{1}{8}$  and  $\log_{27} 9$ .

**Answer:**  $\frac{1}{2}$ ,  $-3$  and  $\frac{2}{3}$  (since  $27^{2/3} = 9$ ).

## 2.10 Inverses of exponential functions

$\log_b x$  undoes  $b^x$ , so its table is the table of  $b^x$  with the rows swapped



The horizontal asymptote  $y = 0$  of  $2^x$  becomes the vertical asymptote  $x = 0$  of  $\log_2 x$ .

| $x$   | 0 | 1 | 2 | 3 | 4  | 5  |
|-------|---|---|---|---|----|----|
| $2^x$ | 1 | 2 | 4 | 8 | 16 | 32 |

For  $\log_2 16$ , find 16 in the  $2^x$  row: the  $x$  above is 4.

| Exponential form | Log form           |
|------------------|--------------------|
| $5^x = 40$       | $x = \log_5 40$    |
| $e^{2t} = 9$     | $2t = \ln 9$       |
| $10^{-2} = 0.01$ | $\log 0.01 = -2$   |
| $y = e^3$        | $\ln y = 3$        |
| $2^{x-1} = 7$    | $x - 1 = \log_2 7$ |

 $y = b^x$  $y = \log_b x$ 

|            |                  |                  |
|------------|------------------|------------------|
| Domain     | all reals        | $x > 0$          |
| Range      | $y > 0$          | all reals        |
| Asymptote  | $y = 0$          | $x = 0$          |
| Key points | $(0, 1), (1, b)$ | $(1, 0), (b, 1)$ |
| Any point  | $(s, t)$         | $(t, s)$         |

## Each undoes the other

$$\begin{aligned} \log_b(b^x) &= x && \text{for every real } x \\ b^{\log_b x} &= x && \text{for } x > 0 \\ \ln e^{5x} &= 5x, \quad 10^{\log 7} = 7, \quad e^{\ln 3} = 3. \end{aligned}$$

## Exam trap

**Writing the inverse of  $b^x$  as  $b^{-x}$  or  $1/b^x$ .** Those reflect the graph across the  $y$ -axis, not  $y = x$ . Since  $2^3 = 8$ , the inverse of  $2^x$  must send 8 back to 3:  $\log_2 8 = 3$  does, but  $2^{-8} = \frac{1}{256}$  does not.

## Worked example

Find the inverse of  $g(x) = 3 \cdot 2^x$ .

- Swap  $x$  and  $y$ :  $x = 3 \cdot 2^y$ , so  $2^y = \frac{x}{3}$ .
- Log form:  $y = \log_2 \frac{x}{3}$ . Check:  $g(3) = 24$  and  $\log_2 \frac{24}{3} = 3$ .

$$g^{-1}(x) = \log_2(x/3), \text{ for } x > 0$$

## Worked example

The point  $(3, 125)$  lies on the graph of  $y = b^x$ . Find  $b$  and a point on  $y = \log_b x$ .

- $b^3 = 125$ , so  $b = 5$ .
- Swap the coordinates for the inverse.

**(125, 3), since  $\log_5 125 = 3$**

## Why a log grows so slowly

Adding 1 to the input of  $2^x$  doubles the output; for  $\log_2 x$  you must double  $x$  to add 1. So  $\log_2 x$  reaches 10 at  $x = 1,024$  but 20 only at  $x \approx 1.05$  million.

## Quick check

$h(x) = 10^x$ . Find  $h^{-1}(1000)$  and  $h^{-1}(0.1)$ .

**Answer:**  $h^{-1}(x) = \log x$ , so 3 and  $-1$ .

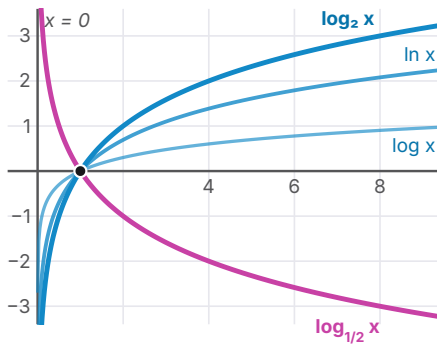
## Logs in other bases

**TI-84 Plus CE.**  $\log$  is base 10 and  $\ln$  is base  $e$ . For  $\log_2 20$ , press  $\text{math}$ , A:  $\log\text{BASE}$ (), and type 2 in the base box and 20 in the brackets, or type  $\ln(20) \div \ln(2)$ . Both give 4.322.

**TI-Nspire CX.**  $\text{ctrl } 10^x$  gives a log with a base box: type 2 there and 20 inside.

**Desmos.** Type  $\log_2(20)$ ; the underscore sets the base.

## 2.11 Logarithmic functions

 $f(x) = a \log_b x$ ,  $a \neq 0$ : domain  $x > 0$ , range all reals, vertical asymptote  $x = 0$ 

Every  $\log_b x$  passes through  $(1, 0)$  and  $(b, 1)$ , and has the  $y$ -axis as its vertical asymptote.

### The domain of a shifted log

$f(x) = a \log_b(x - h) + k$  needs  $x - h > 0$ , so its domain is  $x > h$  and its asymptote is  $x = h$ . Find where the argument is positive before anything else:  $\ln(5 - x)$  needs  $x < 5$ , so its domain lies to the left of its asymptote.

| Case ( $a > 0$ ) | Shape                    | $x \rightarrow 0^+$ | $x \rightarrow \infty$ |
|------------------|--------------------------|---------------------|------------------------|
| $b > 1$          | increasing, concave down | $-\infty$           | $\infty$               |
| $0 < b < 1$      | decreasing, concave up   | $\infty$            | $-\infty$              |

If  $a < 0$ , every entry reverses (increasing and decreasing, concave up and concave down,  $\infty$  and  $-\infty$ ). There is never a maximum, minimum or inflection point.

### Writing the limit at the asymptote

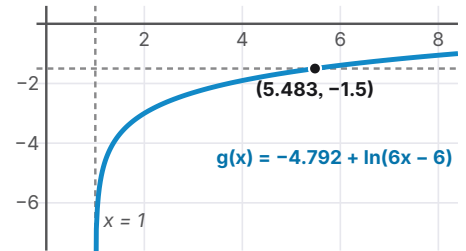
For  $g(x) = -4.792 + \ln(6x - 6)$  as  $x$  decreases toward 1:

$$\lim_{x \rightarrow 1^+} g(x) = -\infty$$

Write  $x \rightarrow 1^+$ : the domain is  $x > 1$ , so  $x$  approaches 1 from values bigger than 1. A plain  $x \rightarrow 1$ , or  $x \rightarrow 1^-$ , does not get the point.

### Spotting a log in a table

If the inputs multiply by a constant factor while the outputs add a constant amount, a log fits. For  $x = 1, 3, 9, 27$  and  $y = 2, 6, 10, 14$ , the inputs triple (base 3), the outputs go up by 4 (multiplier 4), and  $y = 2$  at  $x = 1$ , so  $y = 2 + 4 \log_3 x$ .



The domain is  $x > 1$ , because  $6x - 6$  must be positive. Solving  $g(x) = -1.5$  gives  $\ln(6x - 6) = 3.292$ , so  $x = 1 + e^{3.292}/6 \approx 5.483$ , where the graph meets the dashed line.

### Quick check

$h(x) = 3 - \log_2(4 - 2x)$ . Find its domain, its vertical asymptote and  $\lim_{x \rightarrow 2^-} h(x)$ .

**Answer:**  $4 - 2x > 0$  gives  $x < 2$ , so the asymptote is  $x = 2$ . As  $x \rightarrow 2^-$  the log falls to  $-\infty$ , so 3 minus it grows without bound:  $\lim_{x \rightarrow 2^-} h(x) = \infty$ .

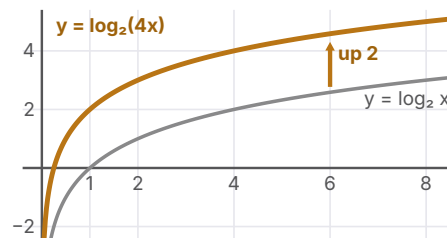
### Exam trap

**Giving a log a horizontal asymptote.** A log looks flat on a screen but never levels off:  $\ln x = 20$  at  $x = e^{20}$ , so  $\lim_{x \rightarrow \infty} \ln x = \infty$ .

## 2.12 Logarithmic function manipulation

A log turns products into sums and powers into multipliers

| Rule, for $M, N > 0$                       | example                                        |
|--------------------------------------------|------------------------------------------------|
| $\log_b(MN) = \log_b M + \log_b N$         | $\log_2(8 \cdot 4) = 3 + 2$                    |
| $\log_b \frac{M}{N} = \log_b M - \log_b N$ | $\log \frac{1000}{10} = 3 - 1$                 |
| $\log_b M^p = p \log_b M$                  | $\log_3 9^4 = 4 \cdot 2 = 8$                   |
| $\log_b M = \frac{\log_a M}{\log_a b}$     | $\log_4 8 = \frac{\ln 8}{\ln 4} = \frac{3}{2}$ |
| $\ln x = \log_e x$                         | $\ln e = 1, \ln 1 = 0$                         |



Since  $\log_2(4x) = \log_2 4 + \log_2 x = 2 + \log_2 x$ , compressing the graph horizontally by 4 lifts it by 2.

### Exam trap

- **Splitting the log of a sum.** No rule does it:  $\log(x + 3)$  is not  $\log x + \log 3$ , which is  $\log 3x$ . At  $x = 7$ :  $\log 10 = 1$ , but  $\log 21 \approx 1.322$ .
- **Moving a power that sits outside the log.**  $\log x^2 = 2 \log x$  for  $x > 0$ , but  $(\log x)^2$  is different: at  $x = 1000$  they are 6 and 9.
- **Turning a quotient of logs into a log of a quotient.**  $\frac{\ln 8}{\ln 2} = \log_2 8 = 3$ , but  $\ln \frac{8}{2} = \ln 4 \approx 1.386$ .
- **Ignoring a change of domain.**  $\ln x^2$  allows every  $x \neq 0$ , but  $2 \ln x$  allows only  $x > 0$ , so they agree only for  $x > 0$ .

### What the rules do to graphs

- By the power rule,  $\log_b x^k = k \log_b x$  for  $x > 0$ : a vertical stretch by  $k$ , with a reflection if  $k < 0$ .
- By change of base, every log is a vertical stretch of every other:  $\log_2 x \approx 1.443 \ln x$ .

### Worked example

Expand  $\log_3 \frac{9x^4}{\sqrt{y}}$  for  $x, y > 0$ .

$$1 \quad \log_3 9 + \log_3 x^4 - \log_3 y^{1/2}$$

$$2 + 4 \log_3 x - \frac{1}{2} \log_3 y$$

### Worked example

No calculator. Write  $2 \log_3 x - \log_3 9 + 1$  as a single logarithm.

- 1 Make every term a log:  $2 \log_3 x = \log_3 x^2$  and  $1 = \log_3 3$ .
- 2 Combine, using the rules in the table:  $\log_3 \frac{x^2 \cdot 3}{9}$ .
- 3 Work out the numbers, since an unsimplified  $\frac{3}{9}$  is not finished.

$$\log_3 \frac{x^2}{3}, \text{ for } x > 0$$

### Worked example

Simplify  $\ln(x^2 - 9) - \ln(x + 3)$  for  $x > 3$ .

- 1 Write it as one log, then factor and cancel:  $\ln \frac{(x-3)(x+3)}{x+3}$ .

$$\ln(x - 3)$$

### Quick check

Write  $\ln 12 - 2 \ln 2$  as one log. Find  $\log 5 + \log 20$ .

**Answer:**  $\ln \frac{12}{2^2} = \ln 3$ ;  $\log 100 = 2$ .

## 2.13

## Exponential and logarithmic equations and inequalities

On the no-calculator question, give exact answers and show every step



| Equation                       | What to do                                                                          | Answer                                          |
|--------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------|
| $2^{x+1} = 8^x$                | same base: $2^{x+1} = 2^{3x}$ , so $x + 1 = 3x$                                     | $x = \frac{1}{2}$                               |
| $3e^{2x} - 4 = 11$             | isolate: $e^{2x} = 5$ , then $2x = \ln 5$                                           | $x = \frac{1}{2} \ln 5$                         |
| $5^x = 12$                     | no common base, so use log form                                                     | $x = \log_5 12 = \frac{\ln 12}{\ln 5}$          |
| $2^x = 3^{x-1}$                | take $\ln$ of both sides: $x \ln 2 = (x - 1) \ln 3$ , so $x(\ln 3 - \ln 2) = \ln 3$ | $x = \frac{\ln 3}{\ln 3 - \ln 2} \approx 2.710$ |
| $\log_2(x + 3) = 4$            | exponential form: $x + 3 = 2^4$                                                     | $x = 13$                                        |
| $\log_2 x + \log_2(x - 2) = 3$ | combine: $x(x - 2) = 8$ , so $x = 4$ or $-2$                                        | $x = 4$ ( $-2$ makes $\log_2 x$ undefined)      |
| $\ln(2x + 1) = \ln(x + 4)$     | equal logs have equal arguments: $2x + 1 = x + 4$                                   | $x = 3$ (both arguments 7)                      |
| $2 \log_3 x = 4$               | $\log_3 x = 2$ ; the domain is $x > 0$                                              | $x = 9$                                         |
| $e^{2x} - e^x - 12 = 0$        | $u = e^x$ : $u^2 - u - 12 = (u - 4)(u + 3)$                                         | $x = \ln 4$ ( $e^x = -3$ is impossible)         |

## Worked example

No calculator. Find every  $x$  with  $e^{2x} - e^x - 12 = 0$ .

- Since  $e^{2x} = (e^x)^2$ , let  $u = e^x$ :  
 $u^2 - u - 12 = 0$ .
- $(u - 4)(u + 3) = 0$ , so  $e^x = 4$  or  $e^x = -3$ . But  $e^x > 0$  for every real  $x$ , so  $e^x = -3$  has no solution.
- Check:  
 $e^{2 \ln 4} - e^{\ln 4} - 12 = 16 - 4 - 12 = 0$ .

**$x = \ln 4$ , the only solution**

## Writing why a root is rejected

Because  $e^x > 0$  for all real  $x$ ,  $e^x = -3$  has no solution.

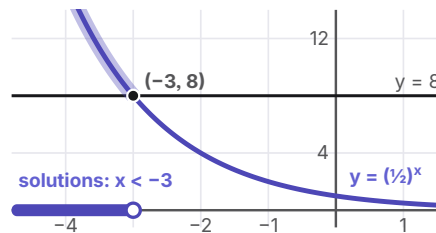
In  $\log_2 x + \log_2(x - 2) = 3$ ,  $x = -2$  is rejected because  $\log_2(-2)$  doesn't exist.

With a calculator: solve  $\ln(6x - 6) - 4.792 = -1.5$ 

**TI-84 Plus CE.** Press  $\boxed{y=}$ , enter  $Y1 = \ln(6x - 6) - 4.792$  and  $Y2 = -1.5$ , then  $\boxed{2nd}$   $\boxed{trace}$  (CALC) 5: intersect,  $\boxed{enter}$ ,  $\boxed{enter}$ , move near the crossing,  $\boxed{enter}$ :  $x \approx 5.483$ .

**TI-Nspire CX.** Graph  $f1(x) = \ln(6x - 6) - 4.792$  and  $f2(x) = -1.5$ .  $\boxed{menu}$   $\rightarrow$  Analyze Graph  $\rightarrow$  Intersection, and click just left of the crossing, then just right of it.

**Desmos.** Type  $y = \ln(6x - 6) - 4.792$  and  $y = -1.5$ , and tap the crossing:  $(5.483, -1.5)$ .



$(1/2)^x$  is decreasing, so it is above 8 (the shaded part of the curve) to the left of  $x = -3$ . Applying  $\log_{1/2}$  to both sides reverses the inequality.

## Solving inequalities

|                        |                                                                     |
|------------------------|---------------------------------------------------------------------|
| $2^x > 10$             | $x > \log_2 10$ (the base is above 1, so the inequality sign stays) |
| $(\frac{1}{2})^x > 8$  | $x < -3$ (the base is below 1, so the inequality sign flips)        |
| $\log_3(x - 1) \leq 2$ | $0 < x - 1 \leq 9$ , so $1 < x \leq 10$                             |
| $\ln x < 0$            | $0 < x < 1$                                                         |

A log inequality also needs its argument to stay positive, which gives a second boundary.

## Changing the base of an exponential

Any exponential can be written in another base:  $b^x = c^{x \log_c b}$ , so  $2^x = e^{x \ln 2} \approx e^{0.693x}$ : the same function written two ways.

## Worked example

No calculator. Solve (a)  $e^{2x} = \frac{1}{e^6}$  and (b)  $\log_2(5x) = 3$ .

a  $e^{2x} = e^{-6}$ , so  $2x = -6$ .

b  $5x = 2^3 = 8$ .

(a)  $x = -3$  (b)  $x = \frac{8}{5}$

## Inverses of shifted exponentials and logs

$f(x) = 4 + 3e^{x-1}$  does four things to  $x$ : subtract 1, raise  $e$  to that power, multiply by 3, add 4. The inverse undoes them in reverse order: subtract 4, divide by 3, take  $\ln$ , add 1.

| $f(x)$                | $f^{-1}(x)$                    |
|-----------------------|--------------------------------|
| $4 + 3e^{x-1}$        | $1 + \ln \frac{x-4}{3}, x > 4$ |
| $2 \log_3(x + 1) - 5$ | $3^{(x+5)/2} - 1$              |
| $ab^{x-h} + k$        | $h + \log_b \frac{x-k}{a}$     |
| $a \log_b(x - h) + k$ | $h + b^{(x-k)/a}$              |

Here  $a \neq 0$ ,  $b > 0$  and  $b \neq 1$ , and the log in the inverse needs  $(x - k)/a > 0$ . Check the first row:  $f(1) = 4 + 3e^0 = 7$  and  $f^{-1}(7) = 1 + \ln 1 = 1$ .

## Exam trap

- Taking the log of each term.** A log can't split a sum, so  $\ln(e^x + 5)$  is not  $x + \ln 5$ . To solve  $e^x + 5 = 12$ , isolate first:  $e^x = 7$ , so  $x = \ln 7$ .
- Treating  $\log_2 x^2 = 4$  and  $2 \log_2 x = 4$  as the same.** In the first,  $x^2 = 16$ , so  $x = \pm 4$ . The second needs  $x > 0$ , so only  $x = 4$ .
- Leaving numbers unevaluated.** Write  $\ln e^{-6}$  as  $-6$  and  $2^3$  as 8.

## Finished exact answers Not finished yet

|                                       |                                 |
|---------------------------------------|---------------------------------|
| $\ln 4, \frac{1}{2} \ln 5$            | $\ln e^{-6}, 3^2$               |
| $\log_5 12$ or $\frac{\ln 12}{\ln 5}$ | $\log_2 8, \log_2(2^3 x)$       |
| $1 + e^2$                             | $\frac{1}{2} \ln \frac{1}{e^6}$ |

## Quick check

Solve  $5^{x+1} = 2$  exactly.

**Answer:**  $x + 1 = \log_5 2$ , so  $x = \log_5 2 - 1$  (or  $\frac{\ln 2}{\ln 5} - 1$ ).

## 2.14 Logarithmic function context and data modeling

### When a log model fits, and how to build one

A log model fits when multiplying the input by the same factor always adds the same amount to the output.

- **From a rate per doubling.** If the output is 0 at  $x = 1$  and rises by 5 each time  $x$  doubles,  $f(x) = 5 \log_2 x$ .
- **From two data points** in a given form, such as  $a + b \ln(t + 1)$ .
- **From data**, by log regression, which gives  $y = a + b \ln x$  (calculator steps on page 10).

### Exam trap

Thinking the formula tells you where the model ends. A log model never levels off:  $R$  keeps falling past 65 and would eventually go below 0. Use the context to decide where the domain ends.

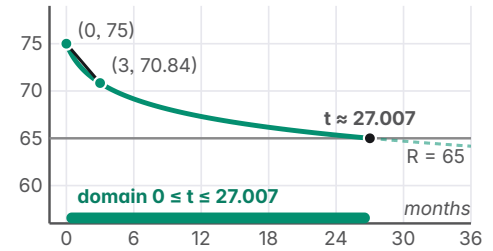
### Worked example

A class averages 75 on a test at  $t = 0$  and 70.84 three months later; the teacher reteaches when the model predicts 65. Fit the class average  $R(t) = a + b \ln(t + 1)$ , find the average rate of change over the 3 months, and find when  $R = 65$ .

- $a + b \ln 1 = 75$  and  $a + b \ln 4 = 70.84$ .
- $\ln 1 = 0$ , so  $a = 75$  and  $b \ln 4 = -4.16$ :  
 $b \approx -3.000806$ .
- Rate on  $[0, 3]$ :  $\frac{70.84 - 75}{3 - 0} \approx -1.387$  points per month.
- $75 + b \ln(t + 1) = 65$  gives  
 $\ln(t + 1) = -10/b$ , so  
 $t = e^{-10/b} - 1 \approx 27.007$ .

$$R(t) \approx 75 - 3.001 \ln(t + 1) \text{ for } 0 \leq t \leq 27.007$$

A log model rises or falls more and more slowly



Average score in points,  $t$  months after the test. The black segment is the secant line for the first 3 months, and the dashed part of the curve is where the model is no longer used.

### Sentences for the rate and the domain

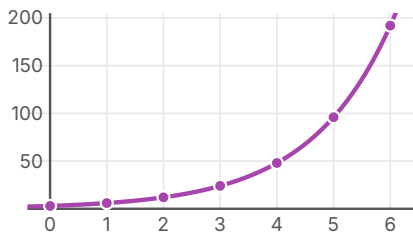
From  $t = 0$  to  $t = 3$  months, the score **decreased on average** by 1.387 points per month.

The model is used until it predicts 65, which happens at  $t \approx 27.007$ , so its domain is  $0 \leq t \leq 27.007$ .

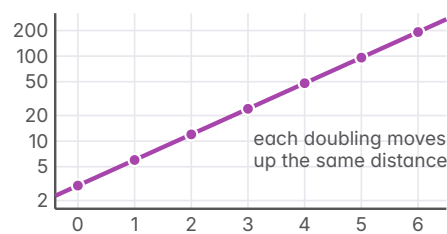
## 2.15 Semi-log plots

A log scale on the y-axis turns exponential data into a line

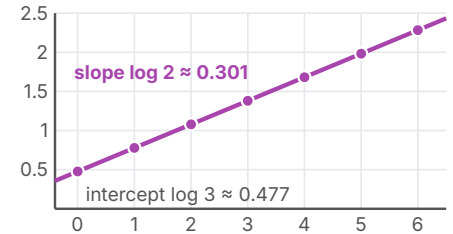
$y = 3 \cdot 2^x$  on an ordinary y-axis



The same data on a log-scaled y-axis



Plotting  $\log_{10} y$  instead of  $y$



### Turning an exponential into a line

Take log of both sides of  $y = ab^x$ :

$$\log y = (\log b)x + \log a$$

slope  $m = \log b$ , so  $b = 10^m$

intercept  $c = \log a$ , so  $a = 10^c$

If the vertical axis shows  $\ln y$ , use base  $e$ :  
 $\ln y = 1.2 + 0.4x$  gives  $a = e^{1.2}$  and  $b = e^{0.4}$ ,  
so  $y \approx 3.320(1.492)^x$ .

### Explaining why an exponential model fits

The semi-log plot is **close to a line**, so  $\log y$  changes at a constant rate and an exponential model fits.

### Exam trap

**Reading the middle of a log axis as the average.** Equal distances on a log axis are equal factors, so halfway between the 10 and 100 marks is  $10^{1.5} \approx 31.6$ , not 55.

### Worked example

A semi-log fit is  $\log_{10} y = 2 + 0.5x$ . Write  $y$  as an exponential.

$$1 \quad y = 10^{2+0.5x} = 10^2 \cdot (10^{0.5})^x$$

$$y = 100(\sqrt{10})^x \approx 100(3.162)^x$$

### Fit the semi-log line to data

**TI-84 Plus CE.** Put  $x$  in L1 and  $y$  in L2. On the L3 heading type  $\log(L2)$  and press **enter**. Then **stat** → **CALC** → 4: **LinReg(ax+b)** with Xlist L1 and Ylist L3. The  $a$  it shows is the slope  $m$  and the  $b$  is the intercept  $c$ ; then  $b = 10^m$  and  $a = 10^c$ .

**TI-Nspire CX.** On a Lists & Spreadsheet page, add a column  $\log y$  with the formula  $\log(y)$ . **menu** → Statistics → Stat Calculations → Linear Regression ( $mx+b$ ), with X List  $x$  and Y List  $\log y$ .

**Desmos.** In a table with columns  $x_1$  and  $y_1$ , type  $\log(y_1) \sim mx_1 + c$ .

### Exam trap

- **Reading the slope as  $b$ .** The slope is  $\log b$ : a slope of 0.5 on a  $\log_{10}$  axis means  $b = 10^{0.5} \approx 3.162$ .
- **Mixing up the two kinds of labels.** A log axis may show the actual values 1, 10, 100 or their logs 0, 1, 2: check which before you compute.
- **Plotting zero or a negative output.** Its log doesn't exist, so leave such points out or use an ordinary axis.
- **Expecting  $ab^x + k$  to plot as a line.** The shift  $k$  bends it; it only looks straight where  $ab^x$  is much bigger than  $k$ .

### Quick check

On a graph of  $\log_{10} y$  against  $x$ , a line passes through  $(0, 0.7)$  and  $(2, 1.3)$ . Find the model.

**Answer:** slope 0.3, so  $b = 10^{0.3} \approx 1.995$  and  $a = 10^{0.7} \approx 5.012$ :  $y \approx 5.012(1.995)^x$ .

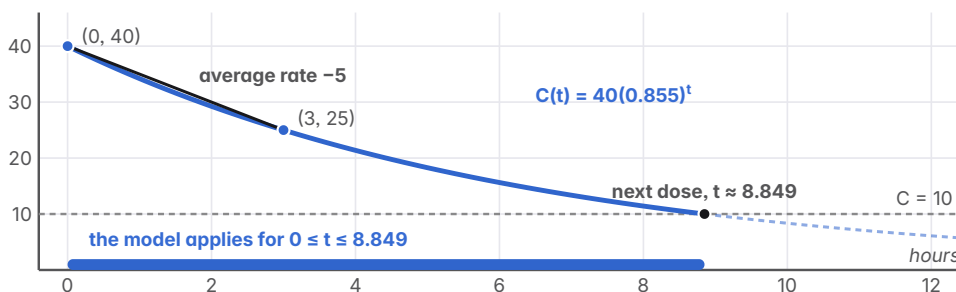
## Free-response questions 1, 2 and 4

Questions 1 and 2 allow a graphing calculator; question 4 does not

## Question 2: building a model from data

In this made-up example, a patient's drug level is 40 mg per liter at  $t = 0$  hours and 25 mg per liter at  $t = 3$ . The model is  $C(t) = ab^t$ , and the next dose is given when the level falls to 10.

| Part                   | An answer that earns credit                                                                                      | What it needs                                                                                   |
|------------------------|------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| (A) equations          | $a = 40$ and $ab^3 = 25$                                                                                         | Write one equation for each data point. A calculator regression alone does not earn this point. |
| (A) values             | $a = 40$ , and dividing gives $b^3 = \frac{25}{40} = 0.625$ , so $b = 0.855$                                     | Give three decimal places, and store $b$ so later parts use every digit.                        |
| (B) average rate       | $\frac{C(3) - C(0)}{3 - 0} = \frac{25 - 40}{3} = -5$                                                             | Write the difference quotient with the numbers put in.                                          |
| (B) meaning            | From $t = 0$ to $t = 3$ hours, the drug level decreased by 5 mg per liter per hour <b>on average</b> .           | Name the interval, the quantity, whether it rose or fell, the units and "on average".           |
| (C) when it reaches 10 | $40b^t = 10$ , so $b^t = 0.25$ and $t = \frac{\ln 0.25}{\ln b} \approx 8.849$ hours                              | Write the equation first, then solve it with logs or the intersect tool.                        |
| (C) domain             | The next dose at $t \approx 8.849$ raises the level again, so the model applies only for $0 \leq t \leq 8.849$ . | Say what happens in the context and give the input where the model stops.                       |



Drug level in mg per liter,  $t$  hours after a dose. The next dose makes the level jump back up, so the model stops at  $t \approx 8.849$ ; the dashed part of the curve is where it no longer applies.

## Small things that cost points

- When a decimal answer is asked for, give three places after the decimal point, rounded or cut off: 8.849, not 8.85. Giving fewer places can cost the point.
- Round only at the end. Rounding  $b$  to 0.85 before solving part (C) gives  $t \approx 8.530$  hours instead of 8.849.
- If a question says "or indicate that it is not defined", that can be the answer:  $f(g(x))$  is not defined when  $g$ 's output is not an input  $f$  accepts.

## Exact answer or decimal?

Question 4 has no calculator, so give exact values such as  $\ln 4$  or  $\log_5 12$ . On questions 1 and 2, give decimals to three places unless the question asks for an exact value.

## Fit an exponential to data

**TI-84 Plus CE.** Press **[stat]**, choose 1: Edit, and type the inputs in L1 and the outputs in L2. Then **[stat]** → **CALC** → 0: ExpReg with Xlist L1 and Ylist L2. It gives  $y = a \cdot b^x$ .

**TI-Nspire CX.** On a Lists & Spreadsheet page, put the data in columns x and y. **[menu]** → Statistics → Stat Calculations → Exponential Regression.

**Desmos.** In a table with columns  $x_1$  and  $y_1$ , type  $y_1 \sim a \cdot b^{(x_1)}$ .

Keep every digit of  $b$ 

**TI-84 Plus CE.** Type  $0.625^{(1/3)}$ , press **[enter]**, then **[sto→]** **[alpha]** **[apps]** **[enter]** saves it as B. Later,  $\ln(0.25) \div \ln(B)$  gives 8.849.

**TI-Nspire CX.** Type  $0.625^{(1/3)}$ , press **[ctrl]** **[var]** (sto→), type b, press **[enter]**. Then  $\ln(0.25)/\ln(b)$  gives 8.849.

**Desmos.** Type  $b = 0.625^{(1/3)}$ , then  $\ln(0.25)/\ln(b)$ . Desmos keeps every digit.

## What question 1 asks from Unit 2

- A composite value from a graph or table and a formula, with the inside value written down. For example, if a graph of  $f$  shows  $f(2) = 5$  and  $g(x) = -4.792 + \ln(6x - 6)$ , write  $g(f(2)) = g(5) = -4.792 + \ln 24 \approx -1.614$ .
- Inputs that give a stated output (inverse values) and solutions of equations, each decimal to three places.
- The limit at a log's asymptote, approached from inside its domain:  $\lim_{x \rightarrow 1^+} g(x) = -\infty$ .
- Whether a function has an inverse, with a reason that uses the function's own values.
- Which type of function fits a table. Equal input steps with a constant ratio mean exponential; inputs that multiply while the outputs add a constant mean logarithmic.

## What question 4 asks from Unit 2

- Solve an exponential or log equation exactly, show the steps, and reject any root outside the domain.
- Rewrite an expression as one log or one power, with the numbers worked out:  $\log_2 x + 3 \log_2 2 = \log_2(8x)$ .
- Its other parts come from Unit 3: trig equations and identities.

## Fit a log model to data

**TI-84 Plus CE.** Press **[stat]**, choose 1: Edit, and type the inputs in L1 and the outputs in L2. Then **[stat]** → **CALC** → 9: LnReg with Xlist L1 and Ylist L2. It gives  $y = a + b \ln x$ .

**TI-Nspire CX.** On a Lists & Spreadsheet page, put the data in columns x and y. **[menu]** → Statistics → Stat Calculations → Logarithmic Regression.

**Desmos.** In a table with columns  $x_1$  and  $y_1$ , type  $y_1 \sim a + b \ln(x_1)$ .

## Check by substituting

Put each answer back into the original equation before moving on. For part (C),  $40(0.855)^{8.849} \approx 10$ , so  $t \approx 8.849$  is right. For  $e^{2x} - e^x = 0$ ,  $x = \ln 4$  gives  $16 - 4 - 12 = 0$ .