

Unit 3

AP Precalculus

Trigonometric and Polar Functions

Radians · Unit circle · Sinusoids · Tangent · Inverse trig · Equations · Identities · Polar

Unit 1

Polynomial and rational functions

30–40%

Unit 2

Exponential and logarithmic functions

25–40%

Unit 3

Trigonometric and polar functions

30–35%

Unit 4

Parameters, vectors and matrices

not tested

Unit 3 is 30–35% of the multiple-choice section. Free-response question 3 is always a periodic model (6 points, no calculator), and question 4 has trig equations and rewrites, also without a calculator.

3.1 Periodic phenomena

3.2 Sine, cosine and tangent

3.3 Unit circle values

3.4 Sine, cosine graphs

3.5 Sinusoidal functions

3.6 Transforming sinusoids

3.7 Sinusoidal models

3.8 The tangent function

3.9 Inverse trig functions

3.10 Equations, inequalities

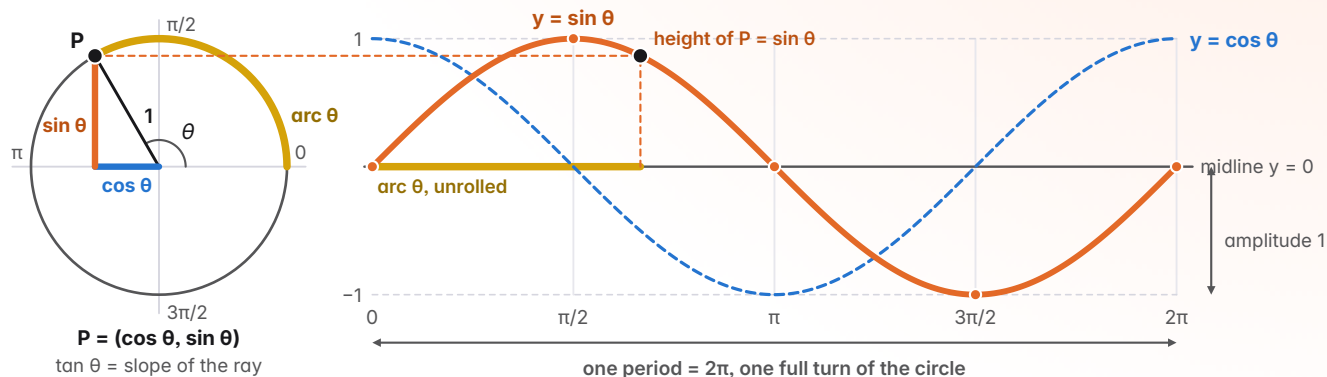
3.11 sec, csc and cot

3.12 Trig identities

3.13 Polar coordinates

3.14 Polar graphs

3.15 Rates of change in polar functions



What the exam asks most

- A sine model built from a story.** The amplitude is half the gap between the max and the min, the midline is halfway between them, and $b = 2\pi/\text{period}$. Then pick c so the model has the right height at the given time: with $a > 0$, $a \sin(b(t + c)) + d$ peaks where $b(t + c) = \frac{\pi}{2}$. (3.6) (3.7)
- Coordinates of five labeled points.** Question 3 shows a graph with five labeled points on one cycle. Their times go up in steps of a quarter period. Starting from a minimum, the heights go min, midline, max, midline, min. (3.7)
- What the function does on an interval.** Say whether the model h is positive or negative (judged from 0, not the midline) and increasing or decreasing, then whether the graph is concave up or down and the rate of change of h is increasing or decreasing. (3.5)
- Exact values without a calculator.** Know sine and cosine at multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, and the one angle each inverse returns. The directions ask for exact values, so write $x = \frac{\pi}{6}$, not 0.524 or an unfinished $\cos^{-1} \frac{\sqrt{3}}{2}$. (3.3) (3.9)
- Every solution in an interval.** For a trig equation, find the angles in one full turn, add whole periods, and list exactly the ones inside the interval. Leaving one out, or adding an extra, loses the point for the full list. (3.10)
- Rewriting an expression in a given form.** Use $\sin^2 x + \cos^2 x = 1$, a double-angle formula or $\sec x = 1/\cos x$, and show every step: the final expression alone earns no credit. (3.12)
- Polar questions, in multiple choice.** Convert between (r, θ) and (x, y) , match $r = f(\theta)$ to its graph, and tell whether the point moves toward or away from the pole (the origin). A negative r points backward, so the distance is $|r|$. (3.13) (3.14) (3.15)

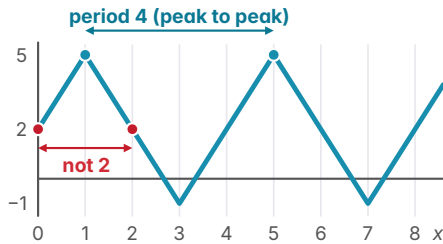
Formulas to know by heart

The exam gives you no formula sheet.

- Radians:** π radians = 180° . Arc length is $s = r\theta$, with θ in radians.
- A point at angle θ** on a circle of radius r is $(r \cos \theta, r \sin \theta)$, and $\tan \theta = \sin \theta / \cos \theta$. In polar form, $x = r \cos \theta$, $y = r \sin \theta$ and $r^2 = x^2 + y^2$.
- For $a \sin(b(\theta + c)) + d$:** amplitude $|a|$, midline $y = d$, period $2\pi/|b|$, and the graph is shifted c units left.
- What each inverse returns:** $\arcsin$ (also written $\sin^{-1}$) gives an angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arccos$ one in $[0, \pi]$, $\arctan$ one in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- Pythagorean identity:** $\sin^2 x + \cos^2 x = 1$. Divide it by $\cos^2 x$ to get $1 + \tan^2 x = \sec^2 x$.
- Sum and difference:** $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$
 $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$ (cosine flips the sign).
- Double angle:** $\sin 2x = 2 \sin x \cos x$, and $\cos 2x = \cos^2 x - \sin^2 x$, which also equals $1 - 2 \sin^2 x$ and $2 \cos^2 x - 1$.

3.1 Periodic phenomena

A periodic function repeats its outputs over equal-length input intervals. The **period** is the smallest positive k with $f(x + k) = f(x)$ for every x in the domain, so one period tells you the whole graph.



The red bracket marks a common mistake. The heights match at 0 and 2, but the graph rises through 2 at $x = 0$ and falls through it at $x = 2$, so 2 is not the period. Peak to peak gives the period, 4.

Everything repeats with the period

Increasing, decreasing, concavity and rates of change all repeat. If f has period 4 and decreases on $(1, 3)$, it also decreases on $(5, 7)$ and on $(-3, -1)$.

Worked example

read a table

$f(t)$ for $t = 0, 1, 2, \dots, 8$ is 2, 5, 2, -1, 2, 5, 2, -1, 2. Assume the pattern keeps repeating. Find the period and $f(41)$.

- The pattern restarts every 4 inputs with the same value *and* the same direction, so the period is 4.
- $41 = 4 \cdot 10 + 1$, so $f(41) = f(1)$.

Period 4, and $f(41) = 5$

Exam trap

- Taking the gap between two equal heights as the period.** In the table, $f(0) = f(2) = 2$, but the outputs after them differ: $f(1) = 5$ while $f(3) = -1$. Check that the whole pattern repeats, not one value. Here it first repeats after 4.
- Giving a longer repeat.** The pattern also repeats every 8 inputs, but the period is the smallest repeat, 4.

Quick check

g has period 6, $g(2) = 5$, and g increases on $(0, 2)$. Find $g(20)$ and an interval after 29 where g increases.

Answer: $20 = 2 + 3 \cdot 6$, so $g(20) = 5$; g increases on $(30, 32)$.

3.2 Sine, cosine, and tangent

Each ratio depends only on the angle

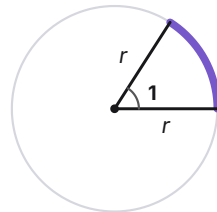
The three definitions

An angle in **standard position** has its vertex at the origin and its initial ray on the positive x -axis. Counterclockwise angles are positive, clockwise ones negative. Where the ray it turns to (the terminal ray) meets a circle of radius r at $P(x, y)$:

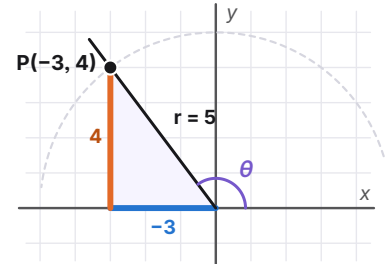
$$\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r} \quad \tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$

Tangent is the slope of the terminal ray. It is undefined when $x = 0$.

If you double the radius, x , y and r all double, so the ratios stay the same. On the unit circle ($r = 1$) they are just the coordinates: $\cos \theta = x$ and $\sin \theta = y$.



1 radian

 $\approx 57.3^\circ$ $\pi \text{ rad} = 180^\circ$ $2\pi \text{ rad} = 360^\circ$ 

Radians

- An angle of θ radians cuts off an arc θ times the radius, so the arc length is $s = r\theta$.
- A full turn is $2\pi \approx 6.28$ radians, or 360° . Multiply by $\pi/180$ to change degrees to radians, and by $180/\pi$ to change back.
- Exam questions use radians unless they say degrees.

degrees	30°	45°	60°	90°	180°	270°
radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

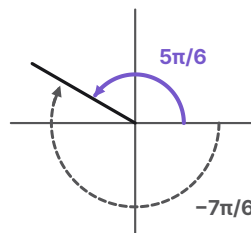
Worked example

arc length and conversion

A wheel of radius 9 inches turns through $\frac{2\pi}{3}$. How far does a point on the rim travel? Convert 225° to radians and $\frac{5\pi}{3}$ to degrees.

- $s = r\theta = 9 \cdot \frac{2\pi}{3} = 6\pi \approx 18.85$ inches
- $225 \cdot \frac{\pi}{180} = \frac{5\pi}{4}$; $\frac{5\pi}{3} \cdot \frac{180}{\pi} = 300^\circ$

6π inches; $\frac{5\pi}{4}$; 300°



Same terminal ray

 $5\pi/6 - 2\pi = -7\pi/6$ counterclockwise: +
clockwise: -they differ by 2π

Coterminal angles share a terminal ray, so they have the same sine, cosine and tangent. Add or subtract 2π until the angle lands in the interval you want.

Worked example

ratios from a point

The terminal ray of θ passes through $P(-3, 4)$. Find $\sin \theta$, $\cos \theta$ and $\tan \theta$.

- $r = \sqrt{(-3)^2 + 4^2} = 5$
- $\sin \theta = \frac{4}{5}$, $\cos \theta = -\frac{3}{5}$, $\tan \theta = -\frac{4}{3}$
- Check the signs: P is in quadrant II, where sine is positive and cosine negative.

$\frac{4}{5}$, $-\frac{3}{5}$, $-\frac{4}{3}$

Exam trap

- Leaving the calculator in degree mode.** In degree mode, $\cos 2$ means the cosine of 2 degrees (0.999), not of 2 radians (-0.416), so every trig answer comes out wrong.
- Putting degrees into $s = r\theta$.** The formula only works in radians, because a radian is defined as arc length divided by radius. The 9-inch wheel turning 120° moves $9 \cdot \frac{2\pi}{3} \approx 18.85$ inches, not $9 \cdot 120 = 1080$.

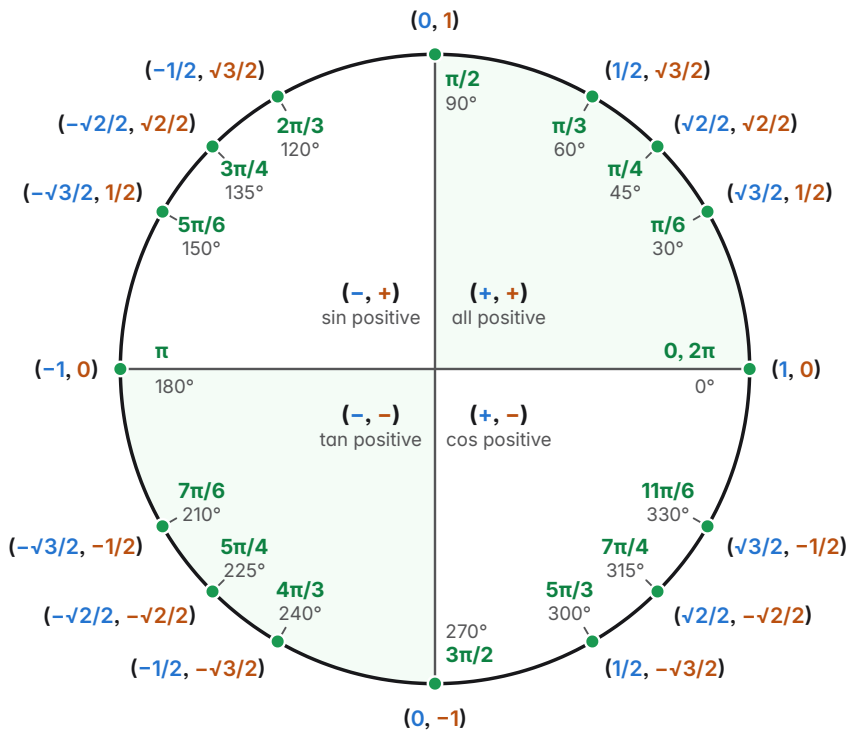
Set radian mode

TI-84 Plus CE. `[mode]`, arrow down to the row Radian Degree, highlight Radian, press `[enter]`, then `[2nd] [mode]` to quit. The top of the screen now says Radian.

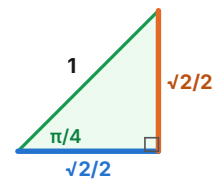
TI-Nspire CX. `[on]` for the home screen, then 5 Settings, 2 Document Settings. Set Angle to Radian, move to Make Default and press `[enter]`.

Desmos. Desmos starts in radians. If the wrench (graph settings) shows Degrees, switch it to Radians.

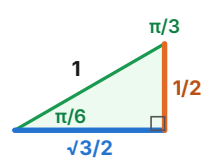
3.3 Sine and cosine values on the unit circle

Every point on the circle of radius 1 is $(\cos \theta, \sin \theta)$ 

half a square



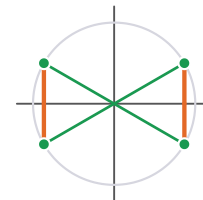
half an equilateral triangle



Each triangle has hypotenuse 1, so its legs are the coordinates: the **blue leg (across)** is the cosine and the **orange leg (up)** is the sine.

Exact values at any other angle

- Reference angle:** the acute angle between the terminal ray and the x-axis. For θ in $[0, 2\pi)$, it is $\pi - \theta$ in II, $\theta - \pi$ in III and $2\pi - \theta$ in IV.
- Size:** the value at the reference angle, ignoring the sign.
- Sign** from the quadrant: sine is positive in quadrants I and II, cosine in I and IV, tangent in I and III.

Reference angle $\pi/6$

$$\sin \pi/6 = 1/2$$

$$\sin 5\pi/6 = 1/2$$

$$\sin 7\pi/6 = -1/2$$

$$\sin 11\pi/6 = -1/2$$

same size, sign by quadrant

■ cosine, the x-coordinate ■ sine, the y-coordinate (-, +): the signs of $(\cos \theta, \sin \theta)$

θ	0	$\pi/6$ (30°)	$\pi/4$ (45°)	$\pi/3$ (60°)	$\pi/2$ (90°)
$\sin \theta$	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2} = 1$
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

From 0 to $\frac{\pi}{2}$, sine runs through $\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}$, each over 2, and cosine runs through the same list backward. Tangent is sine divided by cosine.

Filling in the rest of the circle

- In the first quadrant, the point at $\frac{\pi}{6}$ is low and far to the right, $(\frac{\sqrt{3}}{2}, \frac{1}{2})$; at $\frac{\pi}{4}$ the two coordinates are equal; at $\frac{\pi}{3}$ the point is high and close to the y-axis, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.
- Every other labeled point is one of these reflected across an axis: the same two numbers, with signs set by the quadrant.

Checking a value you remember

Every point on the unit circle satisfies $\cos^2 \theta + \sin^2 \theta = 1$: for $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, $\frac{3}{4} + \frac{1}{4} = 1$. This catches a wrong number but not a swapped pair, so also picture the angle: a small angle has a small height.

Worked example a point on any circle

Find the point 6 m from the origin at angle $\frac{5\pi}{6}$.

$$1 \quad (r \cos \theta, r \sin \theta) = (6 \cdot (-\frac{\sqrt{3}}{2}), 6 \cdot \frac{1}{2})$$

$$(-3\sqrt{3}, 3)$$

Worked example an angle from its values

Which θ in $[0, 2\pi)$ has $\cos \theta = -\frac{1}{2}$ and $\sin \theta < 0$?

- Cosine is $-\frac{1}{2}$ in quadrants II and III, with reference angle $\frac{\pi}{3}$.
- Of those two, sine is negative only in quadrant III, so $\theta = \pi + \frac{\pi}{3}$.

$$\theta = \frac{4\pi}{3}, \text{ the point } (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$$

Worked example no calculator

Find $\sin \frac{7\pi}{6}$, $\cos \frac{3\pi}{4}$, $\tan \frac{5\pi}{3}$, $\cos(-\frac{2\pi}{3})$.

- $\frac{7\pi}{6}$: reference $\frac{\pi}{6}$, quadrant III, sine negative
- $\frac{3\pi}{4}$: reference $\frac{\pi}{4}$, quadrant II, cosine negative
- $\frac{5\pi}{3}$: reference $\frac{\pi}{3}$, quadrant IV, tangent negative
- $-\frac{2\pi}{3}$ ends where $\frac{4\pi}{3}$ does: reference $\frac{\pi}{3}$, quadrant III, cosine negative

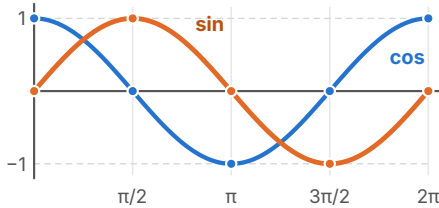
$$-\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{3}, -\frac{1}{2}$$

Exam trap

- Swapping the values at $\frac{\pi}{6}$ and $\frac{\pi}{3}$.** $\sin \frac{\pi}{6} = \frac{1}{2}$, not $\frac{\sqrt{3}}{2}$. $\frac{\pi}{6}$ is a shallow angle, so its height, the sine, is the smaller number.
- Writing $\tan \frac{\pi}{6} = \sqrt{3}$.** Divide: $\frac{1}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. The value $\sqrt{3}$ belongs to the steeper angle, $\frac{\pi}{3}$.
- Giving a decimal when an exact value is asked for.** These questions are on the no-calculator parts and want exact answers, so 0.866 earns no credit; write $\frac{\sqrt{3}}{2}$.

3.4 Sine and cosine graphs

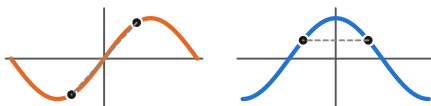
Plot the height of the point P on the unit circle against the angle and you get $y = \sin \theta$; plot its horizontal position and you get $y = \cos \theta$. Both take every value from -1 to 1 as θ runs through all real numbers.



θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	1	0	-1	0
cos	1	0	-1	0	1

Know by heart

- Both have domain all real numbers, range $[-1, 1]$ and period 2π .
- At $\theta = 0$, sine is on the midline going up and cosine is at its maximum.
- Five points a quarter period apart ($0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$) are enough to sketch a whole cycle.



sine is odd

$$\sin(-\theta) = -\sin \theta$$

cosine is even

$$\cos(-\theta) = \cos \theta$$

$$\text{So } \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \text{ and } \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}.$$

Exam trap

- Mixing up the circle and the graph.** On the unit circle the point is $(\cos \theta, \sin \theta)$; on the graph of sine it is $(\theta, \sin \theta)$. At $\theta = \frac{\pi}{2}$ the circle point is $(0, 1)$ and the graph point is $(\frac{\pi}{2}, 1)$.
- Drawing sine with straight lines.** Sine rises fastest at the midline and slowest near the top. In three steps of $\frac{\pi}{6}$ from 0 it gains 0.50, then 0.37, then 0.13, so the peaks are rounded.

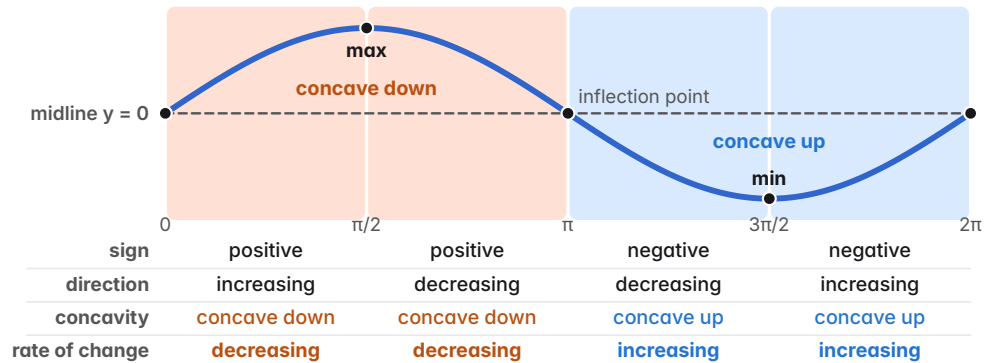
Quick check

On $[0, 2\pi]$: where is $\cos \theta = -1$, and where is $\sin \theta$ increasing?

Answer: $\theta = \pi$; on $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$.

3.5 Sinusoidal functions

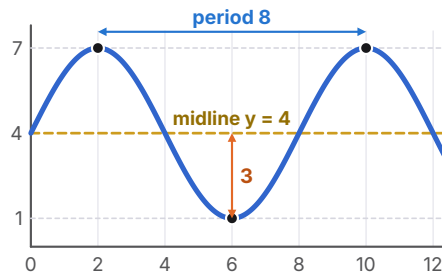
Question 3, part C, asks about one of these quarters



The concavity and rate rows hold for every sinusoid, because the bend always switches at the midline. The sign row holds here only because this midline is $y = 0$.

The words for a sinusoid

sinusoid	a stretched or shifted sine graph; cosine is one too
amplitude	$(\max - \min)/2$, never negative
midline	$y = (\max + \min)/2$
period	the length of one cycle, from one maximum to the next
frequency	$1/\text{period}$, in cycles per unit



Worked example features from a graph

A sinusoid has consecutive maxima of 7 at $t = 2$ and $t = 10$ and a minimum of 1. Find its features.

- Amplitude $\frac{7-1}{2} = 3$; midline $y = \frac{7+1}{2} = 4$
- Period $10 - 2 = 8$, so $b = \frac{2\pi}{8} = \frac{\pi}{4}$; frequency $\frac{1}{8}$ cycle per unit
- The maximum at $t = 2$ is a quarter period after $t = 0$, so the curve starts on the midline going up: no shift.

$$\text{One model: } f(t) = 3 \sin\left(\frac{\pi}{4}t\right) + 4$$

Exam trap

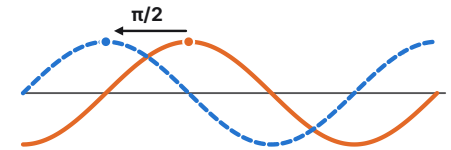
Measuring positive and negative from the midline. Positive or negative means above or below zero; concave up or down is decided by the midline. A point on a tire at height $9 \sin(\pi(t-1)) + 9$ is never below 0, yet on the lower half of its cycle the graph is concave up.

How to word the answer

Write one statement for each:

- " h is negative and increasing" (the function)
- "the graph of h is concave up" (the graph)
- "the rate of change of h is increasing" (the rate)

If the question asks about both concavity and the rate of change, describing only one of them misses the point.

 $\sin \theta$ $\cos \theta = \sin(\theta + \pi/2)$, dashed

Cosine is sine shifted $\frac{\pi}{2}$ to the left: $\cos \theta = \sin(\theta + \frac{\pi}{2})$. So every transformation in Topic 3.6 works the same way on both.

Exam trap

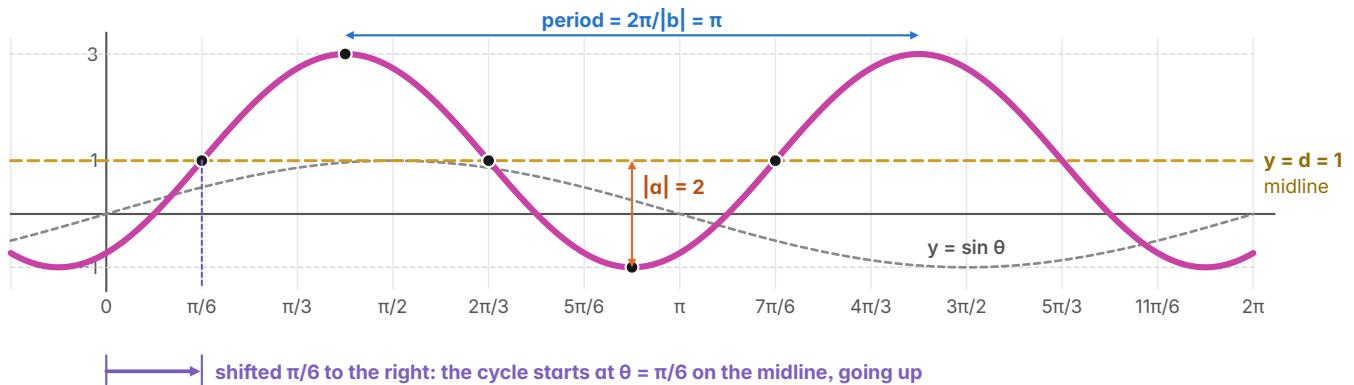
- Taking the maximum as the amplitude.** With a maximum of 7 and a minimum of 1, the amplitude is $\frac{7-1}{2} = 3$, not 7: it is the distance from the midline to a peak.
- Putting the midline at the bottom of the wave.** It is halfway between the maximum and the minimum: $y = 4$ here, not $y = 1$.
- Mixing up period and frequency.** They are reciprocals: a period of $\frac{1}{200}$ second means 200 cycles per second, not $\frac{1}{200}$ of a cycle.
- Thinking "negative and increasing" is impossible.** On $(\frac{3\pi}{2}, 2\pi)$, $\sin \theta$ is negative and increasing as it climbs from its minimum back to 0.

3.6 Sinusoidal function transformations

Numbers outside the sine change the heights; numbers inside change the inputs

$$f(\theta) = a \sin(b(\theta + c)) + d$$

$$\text{Example: } f(\theta) = 2 \sin(2(\theta - \frac{\pi}{6})) + 1$$



what it does

question 3, part B

- a** sets the amplitude $|a|$ (a not 0); $a < 0$ flips the wave
- d** moves the wave and its midline up d
- b** sets the period $\frac{2\pi}{|b|}$ (b not 0)
- c** moves the wave c units left (right if $c < 0$)

Reading the shift when b is not 1

Factor b out first.
 $\sin(2\theta - \frac{\pi}{3}) = \sin(2(\theta - \frac{\pi}{6}))$, so the shift is $\frac{\pi}{6}$ to the right, not $\frac{\pi}{3}$. In general, $a \sin(b\theta + k) + d$ moves $\frac{k}{b}$ units left (right if $\frac{k}{b} < 0$).

Worked example read an equation

Describe $g(\theta) = -3 \sin(2\theta - \frac{\pi}{3}) + 4$.

- Factor: $-3 \sin(2(\theta - \frac{\pi}{6})) + 4$
- Amplitude 3, period $\frac{2\pi}{2} = \pi$, midline $y = 4$, range $[1, 7]$
- Shifted $\frac{\pi}{6}$ right. A quarter period is $\frac{\pi}{4}$; as $a < 0$, it first goes down.

Heights at $\frac{\pi}{6}, \frac{5\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \frac{7\pi}{6}$: 4, 1, 4, 7, 4

Exam trap

- Thinking a bigger b makes a longer period.** The period is $\frac{2\pi}{|b|}$, so a bigger b squeezes the wave: $b = 4$ gives a period of $\frac{\pi}{2}$.
- Shifting the wrong way.** $\sin(\theta - \frac{\pi}{6})$ reaches each height $\frac{\pi}{6}$ later than $\sin \theta$, so its graph moves $\frac{\pi}{6}$ right. Moving it left is the mistake.

A cosine works the same way, but its cycle starts at a maximum (a minimum if $a < 0$).

3.7 Sinusoidal models from context and data

Building a model, step by step

- $d = \frac{\max + \min}{2}$ and $|a| = \frac{\max - \min}{2}$.
- $b = \frac{2\pi}{\text{period}} = 2\pi \times \text{frequency}$.
- Pick sine or cosine and the sign of a from the start at $t = 0$ (below), or use a known point: a sine with $a > 0$ peaks where $b(t + c) = \frac{\pi}{2}$.
- Check two more points a quarter period apart.



Exam trap

Using the frequency as b . A point that vibrates 200 times per second has a period of $\frac{1}{200}$ s, so $b = 2\pi \cdot 200 = 400\pi$, not 200.

For data, use sinusoidal regression. The calculator reports $y = a \sin(bx + c) + d$; there the graph is shifted c/b units left.

Fit a sinusoid to data (radian mode)

TI-84 Plus CE. **[stat]** 1: Edit; inputs in L1, outputs in L2. Then **[stat]** → **CALC** → **C:** SinReg, Xlist L1, Ylist L2, Calculate.

TI-Nspire CX. In Lists & Spreadsheet, put the data in columns named x and y . **[menu]** → **Statistics** → **Stat Calculations** → **Sinusoidal Regression**, X List x , Y List y .

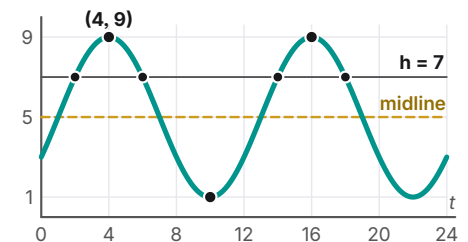
Desmos. In a table with columns x_1 and y_1 , type $y_1 \sim a \cdot \sin(b \cdot x_1 + c) + d$.

Worked example a tide, built by hand

High tide is 9 ft at 4 a.m.; the next low tide is 1 ft at 10 a.m. With t in hours after midnight, write $h(t) = a \sin(b(t + c)) + d$.

- $d = \frac{9+1}{2} = 5$ and $a = \frac{9-1}{2} = 4$. High to low is half a period, so the period is 12 h and $b = \frac{\pi}{6}$.
- Maximum at $t = 4$: $\frac{\pi}{6}(4 + c) = \frac{\pi}{2}$, so $4 + c = 3$ and $c = -1$.
- Check: $h(10) = 4 \sin(\frac{3\pi}{2}) + 5 = 1 \checkmark$

$$h(t) = 4 \sin(\frac{\pi}{6}(t - 1)) + 5$$



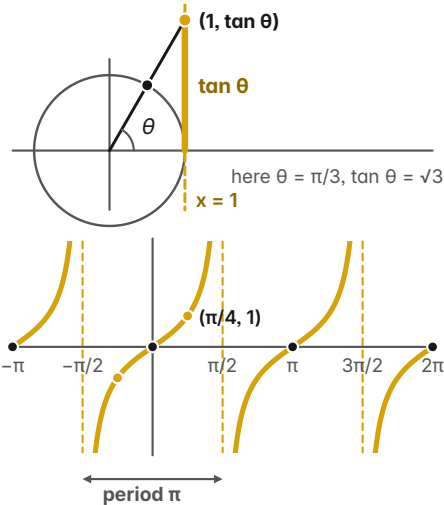
To find when the tide is 7 ft, solve $4 \sin(\frac{\pi}{6}(t - 1)) + 5 = 7$: $\sin(\frac{\pi}{6}(t - 1)) = \frac{1}{2}$ at $t = 2$ (rising) and $t = 6$ (falling), then every 12 hours.

Exam trap

- Giving a cosine when the sine form is asked for.** A correct cosine model like $4 \cos(\frac{\pi}{6}(t - 4)) + 5$ earns only partial credit when the question gives $h(t) = a \sin(b(t + c)) + d$. Convert it: the sine form peaks where $\frac{\pi}{6}(t + c) = \frac{\pi}{2}$, so a peak at $t = 4$ gives $c = -1$.
- Getting the sign of c backwards.** In $b(t + c)$, $c = -1$ delays everything 1 hour: the high tide moves from $t = 3$ (no shift) to $t = 4$. Writing $c = +1$ would put it at $t = 2$.

3.8 The tangent function

$\tan \theta = \frac{\sin \theta}{\cos \theta}$ is the slope of the terminal ray. On the unit circle, extend the ray into a full line: $\tan \theta$ is the height where that line meets $x = 1$.



Know by heart

- The period is π : a ray and the opposite ray have the same slope.
- Zeros at $\theta = k\pi$ (k any integer); vertical asymptotes at $\theta = \frac{\pi}{2} + k\pi$, where $\cos \theta = 0$.
- The range is all real numbers, so tangent has no amplitude.
- It increases between asymptotes: concave down left of each zero, concave up right of it.

Worked example transformed tangent

Find the period, the asymptotes and one point of $y = 2 \tan(3x)$.

- Period $\frac{\pi}{|b|} = \frac{\pi}{3}$
- Asymptotes where $3x = \frac{\pi}{2} + k\pi$ (k any integer), so at $x = \frac{\pi}{6} + \frac{k\pi}{3}$
- At $x = \frac{\pi}{12}$: $2 \tan \frac{\pi}{4} = 2$

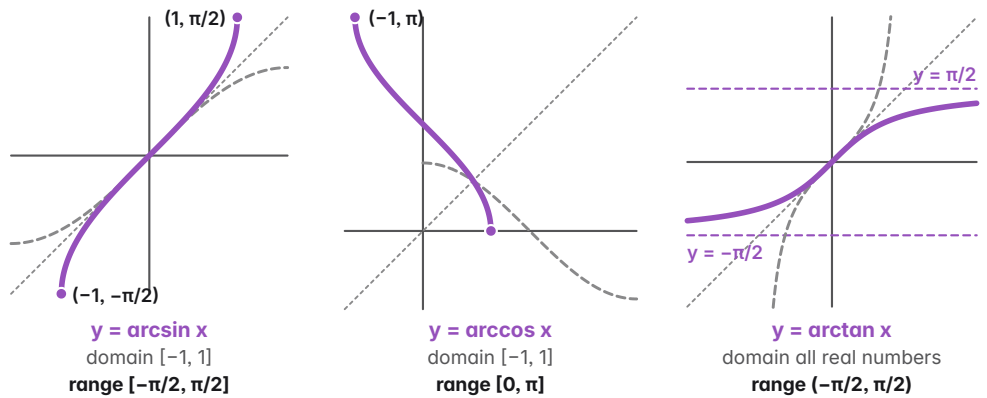
Period $\frac{\pi}{3}$; asymptotes $x = \frac{\pi}{6} + \frac{k\pi}{3}$; through $(\frac{\pi}{12}, 2)$

Exam trap

- Using 2π as tangent's period. Tangent repeats every π , so $\tan(3x)$ repeats every $\frac{\pi}{3}$, not $\frac{2\pi}{3}$.
- Giving $\tan \frac{\pi}{2}$ a value. $\cos \frac{\pi}{2} = 0$, and you can't divide by 0, so it is undefined: not infinity, and not a big number.

3.9 Inverse trigonometric functions

Each inverse gives back one angle

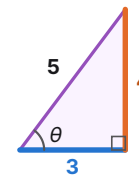


Sine, cosine and tangent repeat, so each is first cut down to a piece that takes every output once (dashed). Reflecting that piece in the line $y = x$ (dotted) gives the inverse, so the piece's inputs become the inverse's outputs. An inverse takes a value and gives back an angle.



The arcs show where each inverse's answers land on the unit circle. The open dots mean arctan never reaches the top or bottom of the circle.

To find $\sin(\arccos \frac{3}{5})$, draw $\theta = \arccos \frac{3}{5}$ in a right triangle.



$\theta = \arccos(3/5)$: $\cos \theta = 3/5$, θ is in $[0, \pi]$; here, quadrant I
opposite = $\sqrt{25 - 9} = 4$
 $\sin(\arccos 3/5) = 4/5$

Worked example undo an inverse

Solve $\arccos(2x - 1) = \frac{\pi}{3}$.

- Take the cosine of both sides: $2x - 1 = \cos \frac{\pi}{3} = \frac{1}{2}$
- $x = \frac{3}{4}$. Check: $\frac{1}{2}$ is in $[-1, 1]$, and $\frac{\pi}{3}$ is in $[0, \pi]$, the range of arccos.

$$x = \frac{3}{4}$$

Exam trap

- Reading $\sin^{-1} x$ as $\frac{1}{\sin x}$. The -1 means the inverse function, arcsine; $\frac{1}{\sin x}$ is $\csc x$. So $\sin^{-1} 0.5 = \frac{\pi}{6} \approx 0.524$, but $\frac{1}{\sin 0.5} \approx 2.086$.
- Assuming $\arcsin(\sin \theta) = \theta$ always. It only holds for θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$: $\arcsin(\sin \frac{5\pi}{6}) = \frac{\pi}{6}$, not $\frac{5\pi}{6}$.
- Assuming $\arccos(-x) = -\arccos x$. That works for arcsin and arctan, but arccos never gives a negative angle. The rule is $\arccos(-x) = \pi - \arccos x$: $\arccos(-\frac{1}{2}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Worked example exact, no calculator

Evaluate $\arcsin(-\frac{1}{2})$, $\arccos(-\frac{1}{2})$, $\arctan(-\sqrt{3})$ and $\arcsin(\sin \frac{5\pi}{6})$.

- arcsin gives an angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so the answer is $-\frac{\pi}{6}$, not $\frac{7\pi}{6}$
- arccos gives an angle in $[0, \pi]$, and cosine is negative there only in quadrant II: $\frac{2\pi}{3}$
- arctan gives an angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$: $-\frac{\pi}{3}$
- $\sin \frac{5\pi}{6} = \frac{1}{2}$, and $\arcsin \frac{1}{2} = \frac{\pi}{6}$

$$-\frac{\pi}{6}, \frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{6}$$

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
arcsin	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
arccos	π	$\frac{5\pi}{6}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	0
x	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
arctan	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

Worked example with a calculator

Solve $\sin x = 0.3$ on $[0, 2\pi)$.

- $\arcsin 0.3 \approx 0.305$ (radian mode), an angle in quadrant I
- Sine is also positive in quadrant II: $\pi - \arcsin 0.3 \approx 2.837$

$$x \approx 0.305 \text{ or } x \approx 2.837$$

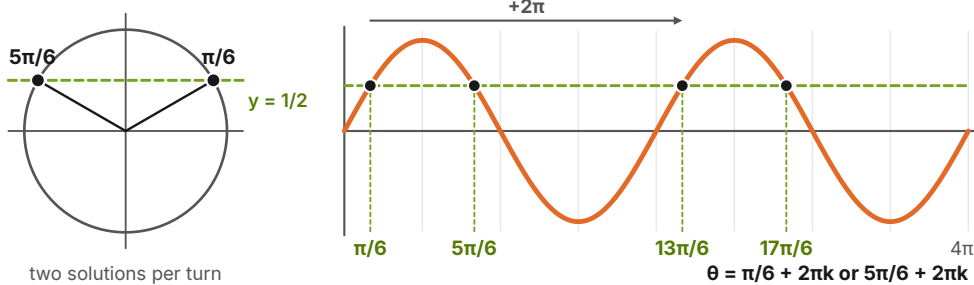
Inverse sine on the calculator

TI-84 Plus CE. Press **2nd** **sin** ($\sin^{-1}$), type 0.3, close the parenthesis and press **enter**.

TI-Nspire CX. Press **trig**, choose $\sin^{-1}$, type 0.3 and press **enter**.

Desmos. Type $\arcsin(0.3)$.

3.10 Trig equations and inequalities



Solving a trig equation

- 1 Isolate** the trig function, for example $\sin u = c$, where u is the angle inside.
- 2 Find the reference angle**, the acute angle for $|c|$, from the unit circle or with $\arcsin |c|$.
- 3 Place it in every quadrant** where the sign of c fits, as on page 3.
- 4 Add whole periods:** $+2\pi k$ for sine and cosine, $+\pi k$ for tangent (k any integer). Keep exactly the answers inside the interval.
- 5 For a multiple angle** such as $3x$, let $u = 3x$. If x runs over $[0, 2\pi)$, u runs over $[0, 6\pi)$: find every u there, then divide each one by 3.

equation **all solutions, with α from the inverse (it may be negative)**

$$\sin u = c \quad u = \alpha + 2\pi k \text{ or } \pi - \alpha + 2\pi k$$

$$\cos u = c \quad u = \pm \alpha + 2\pi k$$

$$\tan u = c \quad u = \alpha + \pi k$$

Sine and cosine equations need $|c| \leq 1$: $\sin x = 2$ has no solution.

Worked example a multiple angle

Solve $2 \cos(2x) + 1 = 0$ on $[0, 2\pi)$.

- $\cos(2x) = -\frac{1}{2}$, and $2x$ runs over $[0, 4\pi)$
- $2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$: two solutions per turn, and $2x$ makes two turns
- Divide each one by 2.

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Worked example a square and a multiple angle

Solve $4 \sin^2(2x) = 3$ on $[0, \pi]$.

- $\sin(2x) = \pm \frac{\sqrt{3}}{2}$, with $2x$ in $[0, 2\pi]$
- $2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$; divide each by 2
- Check one: $4 \sin^2 \frac{\pi}{3} = 4 \cdot \frac{3}{4} = 3 \checkmark$

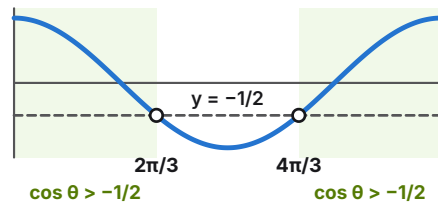
$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$$

Worked example quadratic in sine

Solve $2 \sin^2 x - \sin x - 1 = 0$ on $[0, 2\pi)$.

- Factor: $(2 \sin x + 1)(\sin x - 1) = 0$
- $\sin x = -\frac{1}{2}$ at $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$ (quadrants III and IV); $\sin x = 1$ at $\frac{\pi}{2}$

$$x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$



To solve an inequality, solve the equation for the boundary points, then read the graph. $2 \cos \theta + 1 > 0$ on $[0, 2\pi)$ means $\cos \theta > -\frac{1}{2}$: $[0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi)$. The inequality is strict, so the boundary points are left out.

An inequality in context

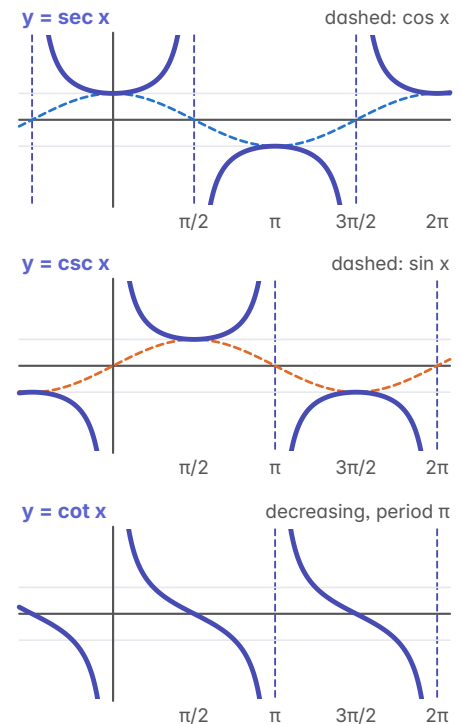
The tide from page 5, $h(t) = 4 \sin(\frac{\pi}{6}(t - 1)) + 5$, is 7 ft at $t = 2$ (rising) and $t = 6$ (falling), so $h(t) \geq 7$ on $[2, 6]$: from 2 a.m. to 6 a.m., and again 12 hours later.

Exam trap

- Dividing both sides by $\sin x$.** That throws away every solution where $\sin x = 0$. In $\sin x \cos x = \sin x$, factor instead: $\sin x(\cos x - 1) = 0$ keeps $x = \pi$ as well as $x = 0$.
- Stopping at one solution, or adding extras.** "Find all x in an interval" needs every solution in it and nothing else. For $2 \cos(2x) + 1 = 0$ on $[0, 2\pi)$, listing only $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ misses two answers, and adding $\frac{7\pi}{3}$, outside the interval, also costs the credit.
- Dropping the minus sign after a square root.** $\tan^2(2x) = 3$ means $\tan(2x) = \sqrt{3}$ or $\tan(2x) = -\sqrt{3}$. Losing the negative case loses half the solutions.

3.11 Secant, cosecant and cotangent

$$\sec x = \frac{1}{\cos x} \quad \csc x = \frac{1}{\sin x} \quad \cot x = \frac{\cos x}{\sin x}$$



Know by heart

- Each has vertical asymptotes where its denominator is 0.
- Secant and cosecant have range $(-\infty, -1] \cup [1, \infty)$ and period 2π .
- Cotangent takes every real value, has period π and decreases between its asymptotes.

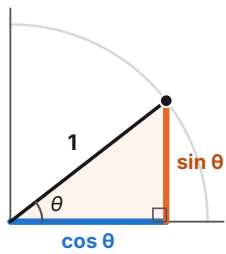
x	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
sec	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
csc	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
cot	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

Exam trap

- Calling $\cot \frac{\pi}{2}$ undefined.** $\cot x = \frac{\cos x}{\sin x}$, which at $\frac{\pi}{2}$ is $\frac{0}{1} = 0$. Writing it as $\frac{1}{\tan x}$ hides this, because $\tan \frac{\pi}{2}$ is undefined.
- Writing $\csc x$ as $\sin^{-1} x$.** Cosecant is $\frac{1}{\sin x}$; $\sin^{-1} x$ is arcsine, a different function.

3.12 Trig identities and equivalent forms

The Pythagorean, sum and difference, and double-angle identities



$$\sin^2 \theta + \cos^2 \theta = 1$$

Pythagorean theorem, hypotenuse 1

 $\div \cos^2 \theta$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

 $\div \sin^2 \theta$

$$1 + \cot^2 \theta = \csc^2 \theta$$

Rearranged forms

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 - \cos^2 \theta = \sin^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\csc^2 \theta - 1 = \cot^2 \theta$$

Sum and difference

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

In the sine formula the sign matches, and each term mixes sine with cosine. In the cosine formula each term pairs like with like, and the sign flips.

Worked example from two known angles

Find $\cos \frac{\pi}{12}$ exactly.

$$\begin{aligned} 1 \quad \frac{\pi}{12} &= \frac{\pi}{3} - \frac{\pi}{4} \\ 2 \quad \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \end{aligned}$$

$$\cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4} \approx 0.966$$

Set $\beta = \alpha$ in the sum formulas and you get the double-angle formulas.

Worked example verify an identity

$$\text{Show that } \frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}.$$

1 Work on the left side only. Multiply its top and bottom by $1 + \cos x$:

$$\frac{\sin x (1 + \cos x)}{1 - \cos^2 x}$$

$$2 \quad 1 - \cos^2 x = \sin^2 x, \text{ so this is } \frac{\sin x (1 + \cos x)}{\sin^2 x} = \frac{1 + \cos x}{\sin x} \checkmark$$

The two sides are equal wherever both are defined

Testing an identity by graphing

On a calculator question, graph each side as its own function. If the graphs differ anywhere, the identity is false. If they match, that supports it, but only the algebra proves it.

Not on the exam

Half-angle identities, and the sum and difference formulas for tangent.

Double angle

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2 \sin^2 x \\ &= 2 \cos^2 x - 1 \end{aligned}$$

Choose the form of $\cos 2x$ that leaves only one function: $1 - 2 \sin^2 x$ when the rest of the problem has sines, $2 \cos^2 x - 1$ when it has cosines.

Worked example from one value

$\sin \theta = \frac{3}{5}$ with θ in quadrant II. Find $\sin 2\theta$ and $\cos 2\theta$.

$$\begin{aligned} 1 \quad \cos \theta &= -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5} \text{ (negative in quadrant II)} \\ 2 \quad \sin 2\theta &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right); \cos 2\theta = 1 - 2\left(\frac{9}{25}\right) \end{aligned}$$

$$\sin 2\theta = -\frac{24}{25}, \cos 2\theta = \frac{7}{25}$$

Worked example solve with an identity

Solve $\cos 2x = \sin x$ on $[0, 2\pi)$.

$$\begin{aligned} 1 \quad &\text{Make it all sine: } 1 - 2 \sin^2 x = \sin x \\ 2 \quad &2 \sin^2 x + \sin x - 1 = 0, \text{ so } (2 \sin x - 1)(\sin x + 1) = 0 \\ 3 \quad &\sin x = \frac{1}{2} \text{ or } \sin x = -1 \end{aligned}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

	$\frac{\pi}{12} (15^\circ)$	$\frac{5\pi}{12} (75^\circ)$
sin	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$
cos	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$

Quick check

Find $\cos \frac{7\pi}{12}$ exactly, and rewrite $\frac{\sec^2 x - 1}{\tan x}$ using only $\tan x$.

$$\text{Answer: } \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}; \frac{\tan^2 x}{\tan x} = \tan x.$$

Rewriting in the form asked for

- Write everything in sine and cosine.
- Look for $1 - \cos^2 x$, $1 - \sin^2 x$ or a double angle.
- Factor, then cancel a factor only where it is not zero.
- Stop at the form asked for: "sin x exactly once" means one sin x and no other trig function.

Show each identity you use; the answer alone earns no credit.

Worked example three rewrites

Simplify each expression.

$$\begin{aligned} 1 \quad &\frac{1 - \cos^2 x}{\sin x \cos x} = \frac{\sin^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} \\ 2 \quad &\sin 2x \csc x = 2 \sin x \cos x \cdot \frac{1}{\sin x} \\ 3 \quad &\text{In } \frac{\cos 2x}{\cos x - \sin x}, \text{ the top } \cos^2 x - \sin^2 x \text{ factors as } (\cos x - \sin x)(\cos x + \sin x); \text{ cancel.} \end{aligned}$$

$$\tan x; 2 \cos x; \cos x + \sin x$$

Exam trap

- Splitting $\sin(\alpha + \beta)$ into $\sin \alpha + \sin \beta$.

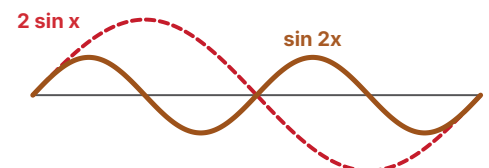
Sine doesn't distribute:

$\sin \frac{\pi}{4} + \sin \frac{\pi}{6} \approx 1.21$, more than any sine can be, while the sum formula gives $\sin(\frac{\pi}{4} + \frac{\pi}{6}) \approx 0.97$.

- Writing $\sqrt{1 - \sin^2 x} = \cos x$. A square root is never negative, so it equals $|\cos x|$. At $x = \pi$ it is 1, but $\cos \pi = -1$.

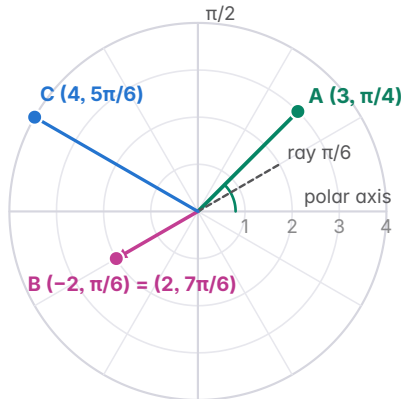
- Losing the original domain. $\sin 2x \csc x$ simplifies to $2 \cos x$, but it is still undefined where $\sin x = 0$, so write "for $\sin x \neq 0$ ".

- Writing $\sin 2x = 2 \sin x$. The 2 inside halves the period and leaves the height alone: at $x = \frac{\pi}{2}$, $\sin 2x = 0$ but $2 \sin x = 2$.



3.13 Polar coordinates

The polar point (r, θ) tells you to turn to angle θ and then go a signed distance r . A negative r goes the same distance the opposite way, through the pole (the origin).



B has a negative r : it turns to $\frac{\pi}{6}$ but lands on the opposite side of the pole, at the same place as $(2, \frac{7\pi}{6})$.

Converting

to (x, y) $x = r \cos \theta$, $y = r \sin \theta$

to (r, θ) $r = \sqrt{x^2 + y^2}$;
 $\theta = \arctan \frac{y}{x}$ if $x > 0$,
 $\theta = \arctan \frac{y}{x} + \pi$ if $x < 0$

If $x = 0$, use $\theta = \frac{\pi}{2}$ when $y > 0$ and $\frac{3\pi}{2}$ when $y < 0$. The pole itself has any angle.

Worked example both directions

Convert $(4, \frac{\pi}{6})$ to (x, y) , and $(-2, 2\sqrt{3})$ to polar.

$$1 \quad (4 \cos \frac{\pi}{6}, 4 \sin \frac{\pi}{6}) = (2\sqrt{3}, 2)$$

$$2 \quad r = \sqrt{4 + 12} = 4. \text{ Since } x < 0, \theta = \arctan(-\sqrt{3}) + \pi = \frac{2\pi}{3}$$

$$(2\sqrt{3}, 2) \text{ and } (4, \frac{2\pi}{3})$$

Different names for the same point

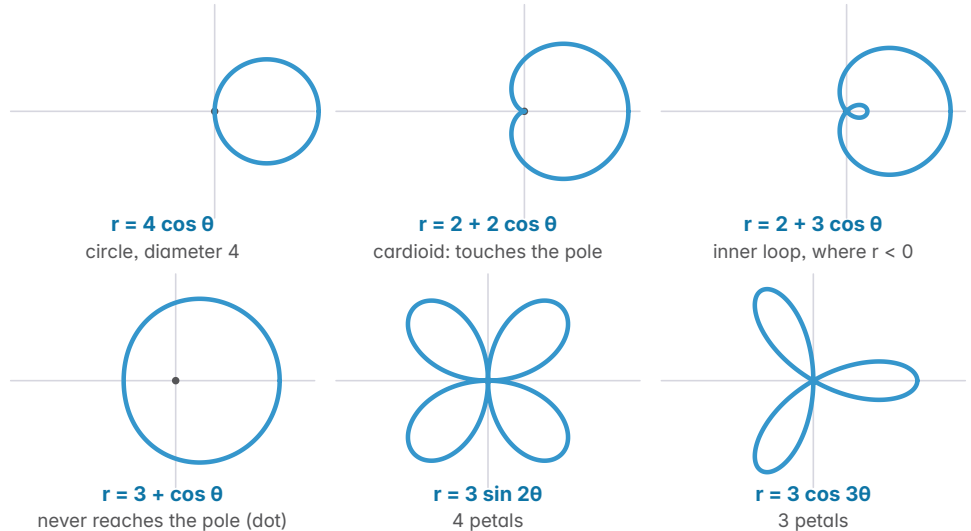
$(r, \theta) = (r, \theta + 2\pi k) = (-r, \theta + \pi + 2\pi k)$,
 so $(3, \frac{\pi}{4}) = (3, \frac{9\pi}{4}) = (-3, \frac{5\pi}{4})$.

Complex numbers use the same plane: $a + bi$ (where $i = \sqrt{-1}$) sits at (a, b) , so $-2 + 2\sqrt{3}i$, at distance 4 and angle $\frac{2\pi}{3}$, can be written $4(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})$.

Exam trap

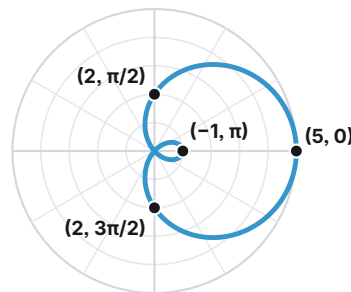
Using $\arctan \frac{y}{x}$ by itself. Arctan only gives angles on the right half of the plane. For $(-1, -1)$, $\arctan 1 = \frac{\pi}{4}$ points into quadrant I, but the point is in quadrant III: add π to get $\frac{5\pi}{4}$.

3.14 Polar function graphs

Each angle θ gets a signed distance r 

Sketching a polar graph

- 1 Make a table of r at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, plus the angles where $r = 0$.
- 2 Plot each point. For a negative r , go the opposite way.
- 3 Join the points in order of increasing θ . That order is the direction the curve is traced.



$$r = 3 \cos \theta + 2$$

Worked example sketch a polar graph

Sketch $r = 3 \cos \theta + 2$ for $0 \leq \theta \leq 2\pi$. Polar points are written (r, θ) .

- 1 At $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$: $r = 5, 2, -1, 2$
- 2 $r < 0$ when $\cos \theta < -\frac{2}{3}$, which makes an inner loop
- 3 Polar $(-1, \pi)$ is $x = 1, y = 0$: the loop crosses the axis at 1

The curve drawn above: it reaches $x = 5$ and its inner loop crosses the axis at $x = 1$.

Polar coordinates from a sensor

A sensor sees an object 14 m away, 0.10 radians from straight ahead:
 $14 \sin 0.10 \approx 1.40$ m to the side and
 $14 \cos 0.10 \approx 13.93$ m ahead.

Recognizing common polar graphs

$r = a \cos \theta$ or $a \sin \theta$	circle through the pole, diameter $ a $
$r = a + b \cos \theta$	$ a = b $: cardioid; $ a < b $: inner loop; $ a > b $: no loop
$r = a \cos(n\theta)$	n petals if n is odd, $2n$ if n is even

Here a and b are not 0 and n is a positive integer. Then check r at a few angles to confirm.

Which angles draw the whole curve

- $r = 4 \cos \theta$ traces its whole circle on $[0, \pi]$; on $[0, 2\pi]$ it goes round twice.
- $r = 3 \sin 2\theta$ repeats its values every π , but needs $[0, 2\pi]$ to draw all four petals.
- A restricted domain such as $0 \leq \theta \leq \frac{\pi}{2}$ draws only that piece.

Exam trap

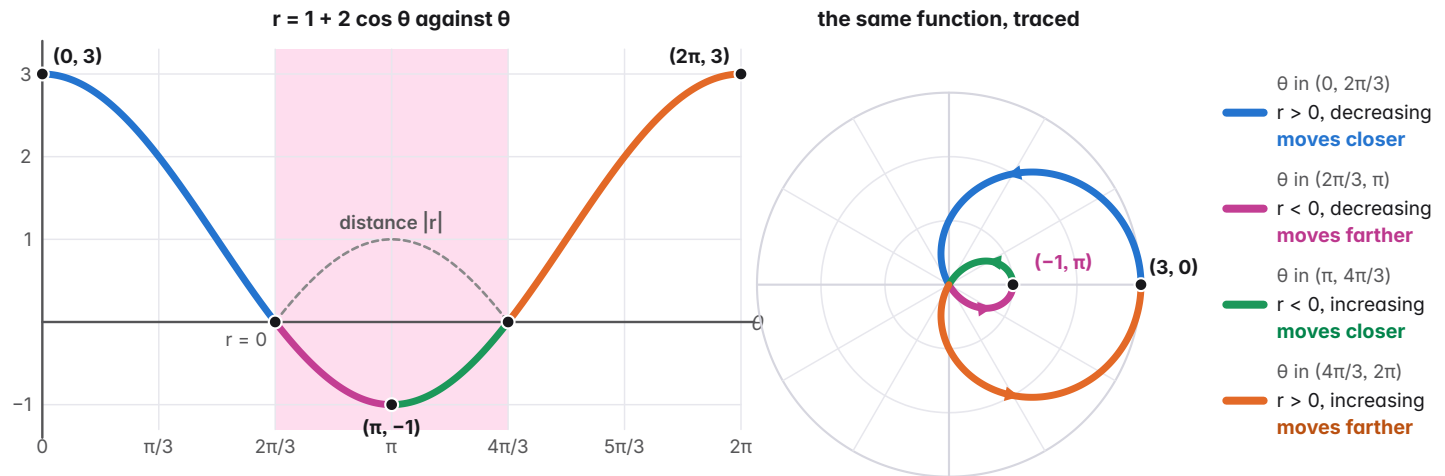
- **Reading a graph of r against θ as the polar curve.** On such a graph (like the one on page 10), the point $(\pi, -1)$ means $\theta = \pi$ and $r = -1$, which is the point $x = 1, y = 0$ in the plane. Convert each pair before you plot it.
- **Skipping points with a negative r .** They are real points, drawn on the opposite side of the pole. For $r = 3 \cos \theta + 2$ they form the inner loop.

Quick check

$r = 2 - 2 \sin \theta$: find r at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and name the shape.

Answer: 2, 0, 2, 4: a cardioid that touches the pole at $\frac{\pi}{2}$ and is farthest from it at the bottom, at $x = 0, y = -4$.

3.15 Rates of change in polar functions

The distance from the pole is $|r|$, not r 

While r is negative (the shaded band, $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$), the point is drawn on the opposite side of the pole and traces the small inner loop. The dashed curve is the distance $|r|$: where r reaches its minimum, -1 , the point is at the loop's **farthest** spot from the pole. Labels on the left graph are (θ, r) ; on the right they are polar points (r, θ) .

Closer to or farther from the pole

	r increasing	r decreasing
$r > 0$	farther	closer
$r < 0$	closer	farther

When r and its change have the same sign (positive and increasing, or negative and decreasing), $|r|$ grows and the point moves away from the pole.

How to word the answer

On $(\frac{2\pi}{3}, \pi)$, r is negative and decreasing, so the distance between the point (r, θ) and the pole is increasing.

Change the underlined words to fit the question. Writing "r is decreasing, so the point gets closer" earns no credit when r is negative.

Where the point is closest or farthest

Where r switches between increasing and decreasing, the point is at its closest or farthest nearby. Use $|r|$ to tell which: at $\theta = \pi$, $r = -1$ is a minimum of r but the farthest point nearby. Where r passes through 0, the curve touches the pole.

The graph of r is a sinusoid

Against θ , $r = 1 + 2 \cos \theta$ is a sinusoid with midline 1 and amplitude 2. Its maximum, 3, is the farthest point, and the stretch below zero is the inner loop.

Average rate of change of r

$$\frac{r(\beta) - r(\alpha)}{\beta - \alpha}$$

This is the change in signed radius per radian over $[\alpha, \beta]$. The slope of the curve is a different quantity.

Worked example rate, then estimate

$r = 1 + 2 \cos \theta$ on $[0, \frac{\pi}{2}]$: find the average rate of change, then use it to estimate $r(\frac{\pi}{4})$.

- $\frac{r(\frac{\pi}{2}) - r(0)}{\frac{\pi}{2} - 0} = \frac{1 - 3}{\pi/2} = -\frac{4}{\pi} \approx -1.27$
- $r(\frac{\pi}{4}) \approx 3 - \frac{4}{\pi} \cdot \frac{\pi}{4} = 2$ (the true value is $1 + \sqrt{2} \approx 2.41$)

$-\frac{4}{\pi}$ units of r per radian; estimate 2

Worked example negative r , growing distance

$r = 2 \cos \theta - 3$ on $[0, \frac{\pi}{2}]$. Is the point moving toward or away from the pole?

- $\cos \theta$ falls on $[0, \frac{\pi}{2}]$, so r falls from -1 to -3 : negative and decreasing
- So the distance $|r|$ grows from 1 to 3.

Away from the pole, even though r decreases

A rate of change of r in context

A machine cuts a spiral groove into a record, its radius dropping 0.12 mm each turn: $\frac{-0.12}{2\pi} \approx -0.019$ mm per radian on average. As r is positive and decreasing, the cutter moves toward the center.

Worked example

read a table

$r = f(\theta)$ changes linearly between these values. Where is the point getting closer to the pole?

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	3	1	-1	-2	-0.5

- r falls from 1 to -1 at a steady rate, so it is 0 halfway, at $\frac{3\pi}{8}$.
- On $(0, \frac{3\pi}{8})$, r is positive and decreasing: closer. On $(\frac{3\pi}{8}, \frac{3\pi}{4})$, negative and decreasing: farther. On $(\frac{3\pi}{4}, \pi)$, negative and increasing: closer.

Closer on $(0, \frac{3\pi}{8})$ and $(\frac{3\pi}{4}, \pi)$

Exam trap

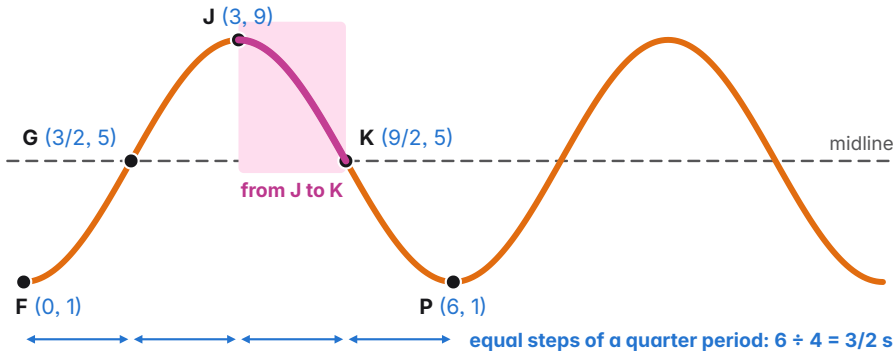
- Taking the smallest r as the closest point.** Distance is $|r|$, so $r = -2$ is 2 units from the pole, farther than $r = 1$.
- Dividing by a change in degrees or turns.** The rate is per radian, so divide by the change in θ in radians. For the groove, -0.12 mm per turn is $-0.12/360$ mm per degree, but per radian it is $-0.12/(2\pi)$.
- Missing a pass through the pole.** If r changes sign on an interval, the distance falls to 0 and then rises. In the table, from $\theta = 0$ to $\frac{\pi}{2}$ the point gets closer, touches the pole at $\frac{3\pi}{8}$, then moves away. Find where $r = 0$ and split the interval there.

Free-response question 3, and the trig parts of question 4

Neither question allows a calculator, and each is worth 6 points

⊗ A question 3 like the exam's, worked through

A paddle wheel of radius 4 ft has its center 5 ft above the water and turns at a constant rate, once every 6 seconds. At $t = 0$ a paddle tip is at its lowest point. $h(t)$ is the tip's height above the water, in feet. **A.** Give coordinates for F, G, J, K and P on the graph. **B.** Find a, b, c, d for $h(t) = a \sin(b(t + c)) + d$. **C.** On the interval from J to K, (t_J, t_K) : (i) is h positive or negative, and increasing or decreasing? (ii) Describe the concavity of the graph and whether the rate of change of h is increasing or decreasing.



Question 3 was scored this way in 2024, 2025 and 2026, one point per row. No work is required for any of these points.

The point	What earns it	What loses it	For the paddle wheel
Part A, the heights	The five heights in the right order, starting wherever the story starts.	Heights out of order, or a wrong midline.	1, 5, 9, 5, 1
Part A, the times	Times a quarter period apart. Any one full cycle works, even one shifted by whole periods.	Steps of a full period, or uneven gaps.	0, $\frac{3}{2}$, 3, $\frac{9}{2}$, 6
Part B, a and d	The amplitude and the midline in the given form. Either sign of a works, as long as c is chosen to match it.	Using the maximum as a , or the minimum as d .	$a = 4, d = 5$
Part B, b and c	$b = 2\pi/\text{period}$, and a c that puts the starting point in the right place. This is the point students miss most often.	Using the frequency as b , the wrong sign of c , or a cosine when the sine form was asked for.	$b = \frac{\pi}{3}, c = -\frac{3}{2}$
Part C (i), sign and direction	The right pair: positive or negative, and increasing or decreasing.	Judging the sign from the midline instead of from 0.	positive and decreasing
Part C (ii), concavity and rate	The concavity <i>and</i> the direction of the rate of change, both correct, with nothing wrong added.	Only one of the two, or saying the rate is "increasing at a decreasing rate", an extra claim that needs calculus to support.	concave down; rate of change decreasing

✍ The trig parts of question 4

- **Solving with an inverse.** $\arcsin \frac{x}{2} = \frac{\pi}{4}$ means $\frac{x}{2} = \sin \frac{\pi}{4}$, so $x = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$. Undo the inverse with its own function, then check that the input is allowed: $\frac{x}{2}$ must be in $[-1, 1]$.
- **Rewriting in the form asked for.** Use one identity at a time and show each step. The final expression alone does not earn the point.
- **All solutions in an interval.** One correct value with its work earns the first point. Every value in the interval, with nothing extra, earns the second; a general formula alone does not.

Worked example

every solution in an interval

Solve $\tan^2(2x) = 3$ for $0 \leq x \leq \frac{\pi}{2}$.

- 1 $\tan(2x) = \sqrt{3}$ or $\tan(2x) = -\sqrt{3}$, with $2x$ in $[0, \pi]$.
- 2 $\tan(2x) = \sqrt{3}$ at $2x = \frac{\pi}{3}$; $\tan(2x) = -\sqrt{3}$ at $2x = \frac{2\pi}{3}$.
- 3 Divide by 2: $x = \frac{\pi}{6}$ or $x = \frac{\pi}{3}$.
- 4 Check: $\tan^2 \frac{\pi}{3} = 3$ and $\tan^2 \frac{2\pi}{3} = 3$ ✓

$$x = \frac{\pi}{6}, \frac{\pi}{3}$$

Writing only $x = \frac{\pi}{6}$ would earn one of the two points. Adding $x = \frac{2\pi}{3}$ (the value of $2x$, not x , and outside the interval) would lose the second.

⊗ A full-credit answer

A F (0, 1), G ($\frac{3}{2}$, 5), J (3, 9), K ($\frac{9}{2}$, 5), P (6, 1)

B $a = 4$ and $d = 5$. The period is 6, so $b = \frac{2\pi}{6} = \frac{\pi}{3}$. The tip is lowest at $t = 0$, and sine is lowest when its input is $-\frac{\pi}{2}$, so $\frac{\pi}{3}(0 + c) = -\frac{\pi}{2}$ and $c = -\frac{3}{2}$.
 $h(t) = 4 \sin(\frac{\pi}{3}(t - \frac{3}{2})) + 5$. ($a = -4$ with $c = \frac{3}{2}$ also works.)

C (i) h is positive and decreasing. (ii) The graph of h is concave down, and the rate of change of h is decreasing.

Check part B against part A: $h(0) = 4 \sin(-\frac{\pi}{2}) + 5 = 1$ ✓ and $h(3) = 4 \sin \frac{\pi}{2} + 5 = 9$ ✓.

Worked example

a rewrite

Rewrite $\frac{\sin 2x}{2 \cos^2 x}$ with $\tan x$ exactly once.

- 1 $\sin 2x = 2 \sin x \cos x$ (double angle)
- 2 Cancel $2 \cos x$, which is not 0 where the expression is defined:

$$\frac{2 \sin x \cos x}{2 \cos^2 x} = \frac{\sin x}{\cos x}$$

$\tan x$

What the question 4 directions ask for

Angles are in radians. Give exact values ($x = \frac{\pi}{6}$, not $\cos^{-1} \frac{\sqrt{3}}{2}$ or 0.524), and show the work that leads to each answer.