

Unit 4

AP Precalculus

Functions Involving Parameters, Vectors, and Matrices

Parametric motion · Implicit curves · Conics · Vectors · Matrices · Transition models

Unit 1

30–40%

Polynomial and rational functions

Unit 2

25–40%

Exponential and logarithmic functions

Unit 3

30–35%

Trigonometric and polar functions

Unit 4

not tested

Parameters, vectors and matrices

Unit 4 is not tested on the AP exam. The percentages are the shares of Units 1–3 in the exam's multiple-choice section. Unit 4 can still be on class tests, and its parametric equations and vectors return in AP Calculus BC.

4.1

Parametric functions

4.2

Planar motion

4.3

Rates of change

4.4

Circles and lines

4.5

Implicit functions

4.6

Conic sections

4.7

Parametrizing curves

4.8

Vectors

4.9

Vector-valued functions

4.10

Matrices

4.11

Inverse and determinant

4.12

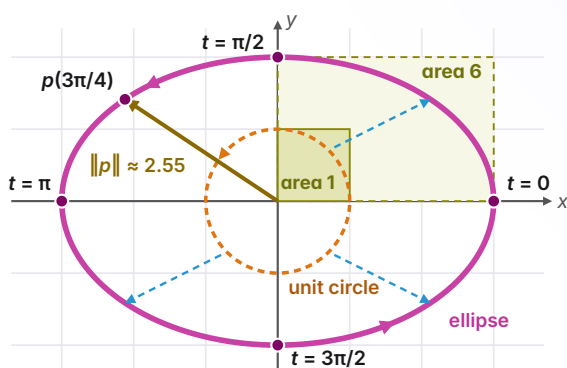
Linear transformations

4.13

Matrices as functions

4.14

Matrix models



The same value of t gives the same point in all four boxes above.

A path traced by a parameter 4.4 4.7

$x = 3 \cos t$, $y = 2 \sin t$ for $0 \leq t \leq 2\pi$ goes once around counterclockwise, starting at $(3, 0)$.

An implicit equation 4.5 4.6

$\frac{x^2}{9} + \frac{y^2}{4} = 1$. Solved for y , it splits into two functions, $y = \pm 2\sqrt{1 - x^2/9}$: the top half and the bottom half.

A moving vector 4.8 4.9

$p(t) = \langle 3 \cos t, 2 \sin t \rangle$ points from the origin to the ellipse. Its length $\|p(t)\|$ is the distance from the origin: 3 at $t = 0$, 2 at $t = \pi/2$.

A matrix acting on the unit circle 4.12 4.13

moves each circle point to the ellipse point with the same t (dashed arrows). It makes widths 3 times and heights 2 times bigger, so the unit square becomes a 3 by 2 rectangle of area 6, and 6 is the determinant.

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

What a class test on Unit 4 asks you to do

- Sketch a parametric path.** Plot the points (x, y) from a table of t values, join them in order of increasing t , and add arrows for the direction and a dot at each end. 4.1
- Describe the motion.** A particle moves right while $x(t)$ increases and up while $y(t)$ increases. Its farthest-right and highest points usually come at different times. 4.2 4.3
- Write circles, segments and conics.** Complete the square to find a conic's center, then write its equation or parametrize it. 4.4 4.6 4.7

- Work with vectors.** A vector's components are its end point minus its start point, and $\cos \theta = u \cdot v / (\|u\| \|v\|)$ gives the angle between two vectors. 4.8 4.9
- Use a 2×2 matrix as a function.** Its columns are where $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ land. The inverse undoes it, and multiplying by it n times runs a transition model n steps. 4.10 4.11 4.13 4.14

Not part of Unit 4

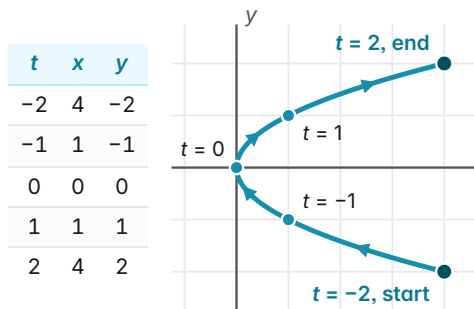
Tilted conics, foci, directrices and eccentricity are not in the course. Vectors stay in two dimensions, inverses stay 2×2 , and you never differentiate $x(t)$ or $y(t)$.

Formulas to know

- Circle**, center (h, k) , radius r : $(h + r \cos t, k + r \sin t)$, $0 \leq t \leq 2\pi$.
- Segment PQ**: $P + t(Q - P)$, $0 \leq t \leq 1$.
- Ellipse**, center (h, k) , reaching a to each side and b up and down: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
- Hyperbola** opening left and right: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, with asymptotes $y - k = \pm \frac{b}{a}(x - h)$.
- Length of $\langle a, b \rangle$** : $\|\langle a, b \rangle\| = \sqrt{a^2 + b^2}$.
- Dot product**: $u_1 v_1 + u_2 v_2 = \|u\| \|v\| \cos \theta$.
- Determinant** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. If it isn't 0, the **inverse** is $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- Transition model** with matrix M : n steps after the state s_0 , the state is $M^n s_0$.

4.1 Parametric functions

A parametric function has one input and two outputs: it sends each value of the parameter t to a point $(x(t), y(t))$.



$x = t^2$, $y = t$ for $-2 \leq t \leq 2$. As a curve in the plane it fails the vertical-line test, but it is still a function, because its input is t , not x .

How to sketch a parametric path

- For each value of t in a table, find x and y from that same t .
- Plot the points (x, y) , not (t, x) or (t, y) .
- Join the points in order of increasing t .
- Add arrows for the direction, and dots where the interval for t starts and stops.

Test trap

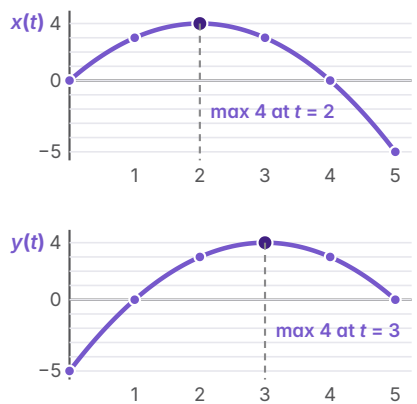
- Joining the points from left to right.** The path goes in order of t , not x . Above, the path comes in from $(4, -2)$ to $(0, 0)$ and then goes back out to $(4, 2)$. Joining the points in order of x would draw a zigzag through $(1, -1)$ and $(1, 1)$ that the particle never makes.
- Drawing more than the interval allows.** The interval for t decides how much of the curve you draw. $(\cos t, \sin t)$ for $0 \leq t \leq \pi$ is only the top half of the unit circle, from $(1, 0)$ to $(-1, 0)$; the whole circle needs $0 \leq t \leq 2\pi$.
- Eliminating t and drawing only $x = y^2$.** Getting rid of t in $x = t^2$, $y = t$ gives $x = y^2$, which has the same shape but no direction and no ends. Keep the arrows and end dots that come from t .

Quick check

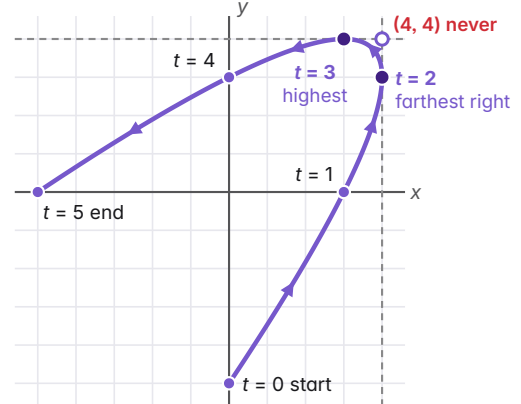
Does (t^2, t) pass through $(4, 2)$? Through $(4, 1)$?

Answer: $(4, 2)$ yes: $y = 2$ forces $t = 2$, and then $x = 4$. $(4, 1)$ no: $y = 1$ forces $t = 1$, but then $x = 1$, not 4.

4.2 Parametric functions modeling planar motion



A particle moves as $p(t) = (4 - (t - 2)^2, 4 - (t - 3)^2)$ for $0 \leq t \leq 5$. The two graphs on the left show how far right it is and how high it is at each time; the graph on the right is its path.



To find	Do this
Farthest right or left	Find the max or min of $x(t)$, and check both ends of the interval.
Highest or lowest	Find the max or min of $y(t)$, and check both ends.
Crossing the y-axis	Solve $x(t) = 0$, then put each t into $y(t)$.
Crossing the x-axis	Solve $y(t) = 0$, then put each t into $x(t)$.
Passing the origin	Solve $x(t) = 0$, then keep only the t values that also make $y(t) = 0$.

Worked example

For the motion in the figure, find the farthest-right point, the highest point and every point where the path meets an axis.

- $x(t) = 4 - (t - 2)^2$ is largest at $t = 2$, so the farthest-right point is $p(2) = (4, 3)$.
- $y(t) = 4 - (t - 3)^2$ is largest at $t = 3$, so the highest point is $p(3) = (3, 4)$.
- $x = 0$ when $(t - 2)^2 = 4$, so $t = 0$ or 4 : the points $(0, -5)$ and $(0, 3)$.
- $y = 0$ when $(t - 3)^2 = 4$, so $t = 1$ or 5 : the points $(3, 0)$ and $(-5, 0)$.

Farthest right $(4, 3)$, highest $(3, 4)$; it meets the axes at $(0, -5)$, $(0, 3)$, $(3, 0)$ and $(-5, 0)$

Quick check

$A(t) = (t, t)$ and $B(t) = (5 - t, 2)$. Their paths cross at $(2, 2)$. Do they collide?

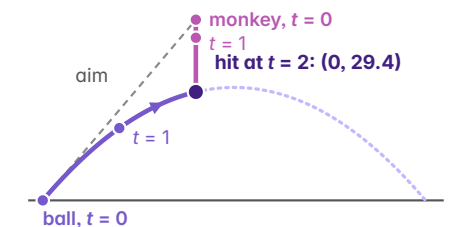
Answer: No. A is at $(2, 2)$ when $t = 2$, but B gets there at $t = 3$. Their x -values match only when $t = 5 - t$, at $t = 2.5$, and then A is at height 2.5 while B is at height 2.

Test trap

- Mixing up which zero gives which axis.** Every point on the y -axis has $x = 0$, so solving $x(t) = 0$ finds the y -axis crossings. In the example, $t = 0$ and $t = 4$ give $(0, -5)$ and $(0, 3)$.
- Building one point from two extremes.** The largest x and the largest y come at different times, so the particle never reaches $(4, 4)$. Find the time of each extreme, then the whole point at that time.
- Answering "where" with a time.** "When" asks for t ; "where" asks for a point. For "where does it cross the x -axis?", $t = 1$ is not the answer; $p(1) = (3, 0)$ is.

Crossing paths and collisions

Two objects collide only if they are at the same point **at the same time**. Set $x_1(t) = x_2(t)$, then check whether y_1 and y_2 agree at that same t .

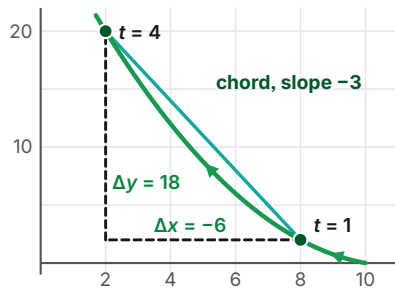


The monkey and the hunter. A hunter 42 m from a tree aims at a monkey 49 m up, and the monkey drops the moment the gun fires. In meters, the ball moves as $(21t - 4.9t^2, 24.5t - 4.9t^2)$ and the monkey as $(0, 49 - 4.9t^2)$. At $t = 2$ s the ball reaches the tree and both are 29.4 m high, so it hits: gravity pulls both down by the same $4.9t^2$ (ignoring air resistance). The faint arc is where the ball would have gone.

4.3 Parametric functions and rates of change

A particle moves right while $x(t)$ is increasing and up while $y(t)$ is increasing.

	$y(t)$ increasing	$y(t)$ decreasing
$x(t)$ increasing	↗ right and up	↘ right and down
$x(t)$ decreasing	↖ left and up	↙ left and down

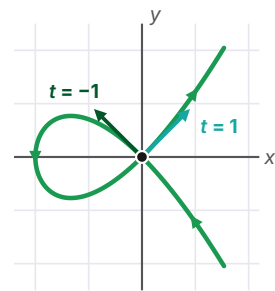


Average rates over an interval

Each coordinate has its own average rate, $\Delta x/\Delta t$ and $\Delta y/\Delta t$. Their quotient is the slope of the chord (the teal line joining the two points), if $\Delta x \neq 0$:

$$\frac{\Delta y/\Delta t}{\Delta x/\Delta t} = \frac{\Delta y}{\Delta x}$$

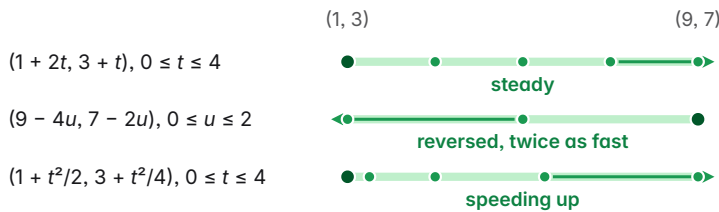
For $x = 10 - 2t$, $y = t^2 + t$ from $t = 1$ to $t = 4$, the rates are -2 and 6 , so the slope is -3 .



The same point at two different times

$x = t^2 - 1$, $y = t^3 - t$ crosses itself at $(0, 0)$, so that one point has two times, $t = -1$ and $t = 1$, and a different direction at each. From $t = -1.1$ to -0.9 , x falls (0.21 to -0.19) and y rises (-0.23 to 0.17), so it moves **left and up**. From $t = 0.9$ to 1.1 both rise: **right and up**.

One segment, three parametrizations



The big dot is the start and the dark arrow at the end the direction. The dots are one unit of the parameter apart, so wider gaps mean faster motion.

⚠ Test trap

- **Dividing coordinates instead of changes.** The slope is the change in y over the change in x , $\frac{18}{6} = -3$ above. Dividing the end coordinates, $\frac{20}{2} = 10$, means nothing.
- **Reading direction from a coordinate's sign.** $x(t) > 0$ only says the particle is right of the y -axis. With $x = 10 - 2t$, $x > 0$ at $t = 1$, yet the particle moves left because x is decreasing.

🔍 Quick check

$p(t) = (t^2 - 2t, 9 - t^2)$. Over $1 \leq t \leq 3$, find both average rates and the chord's slope. Which way is the particle moving?

Answer: x goes from -1 to 3 (rate 2) and y from 8 to 0 (rate -4), so the slope is -2 . On the whole interval x increases and y decreases: right and down.

4.4 Parametrically defined circles and lines

📖 Writing a circle

$(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ goes once counterclockwise around the unit circle from $(1, 0)$. For center (h, k) and radius r , write $x = h + r \cos \theta$, $y = k + r \sin \theta$, then choose the angle θ , measured from the positive x -axis:

One lap in T seconds

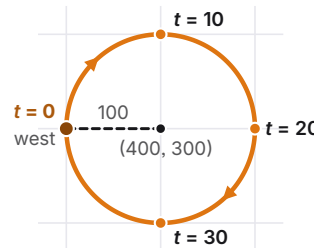
$$\theta = \frac{2\pi}{T}t, \text{ for } 0 \leq t \leq T$$

Start at angle φ

$$\theta = \varphi + \frac{2\pi}{T}t$$

Clockwise

$$\theta = \varphi - \frac{2\pi}{T}t$$

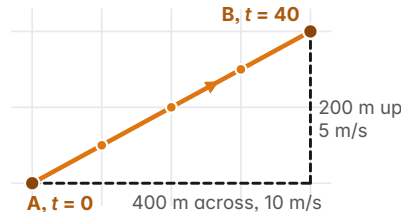


📖 Worked example

A drone circles a fire at $(400, 300)$, 100 m from it, **clockwise**, starting **due west** of it, one lap every 40 s.

- 1 West is angle π , and going clockwise makes it shrink: $\theta = \pi - \frac{2\pi}{40}t = \pi - \frac{\pi}{20}t$.
- 2 Check: $t = 10$ gives $\theta = \frac{\pi}{2}$, due north ✓

$$x = 400 + 100 \cos\left(\pi - \frac{\pi}{20}t\right), \\ y = 300 + 100 \sin\left(\pi - \frac{\pi}{20}t\right), \\ 0 \leq t \leq 40$$



📖 Segments at a steady speed

From (x_1, y_1) to (x_2, y_2) in T seconds:

$$x = x_1 + \frac{x_2 - x_1}{T}t \\ y = y_1 + \frac{y_2 - y_1}{T}t \\ \text{for } 0 \leq t \leq T.$$

📖 Worked example

The drone flies from $A(-100, 100)$ to $B(300, 300)$ in 40 s at a steady speed. Write its path.

- 1 Rates: $\frac{300 - (-100)}{40} = 10$ m/s across and $\frac{300 - 100}{40} = 5$ m/s up.

$$x = -100 + 10t, y = 100 + 5t, 0 \leq t \leq 40; \text{ at } t = 40 \text{ it is at } (300, 300) \checkmark$$

⚠ Test trap

- **Putting the distance in front of t .** The number in front of t is a rate: 400 m in 40 s is 10 m/s, so $x = -100 + 10t$. Writing 400t would put it at $x = 15,900$ when $t = 40$.
- **Swapping sine and cosine.** $(\sin t, \cos t)$ starts at the top, $(0, 1)$, and runs clockwise: at $t = \frac{\pi}{2}$ it is at $(1, 0)$. Put in $t = 0$ and $t = 0.1$ to see the start and the direction.

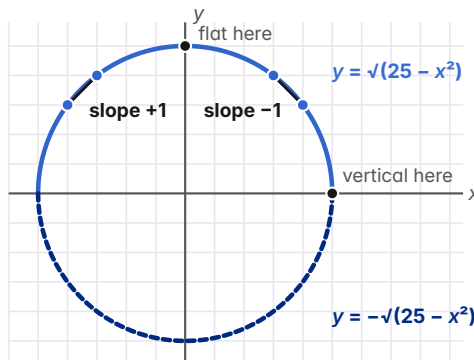
🔍 Quick check

Parametrize $(x - 3)^2 + (y + 2)^2 = 25$, once around counterclockwise, starting at $(8, -2)$.

Answer: $x = 3 + 5 \cos t$, $y = -2 + 5 \sin t$, $0 \leq t \leq 2\pi$; $t = 0$ gives $(8, -2)$ ✓

4.5 Implicitly defined functions

Solve an equation in x and y for one variable, and each function you get draws part of the curve.



$x^2 + y^2 = 25$ splits into two functions of x , the top and bottom halves, each with domain $-5 \leq x \leq 5$.

Equation	Solved for one variable
$x^2 + y^2 = 25$	$y = \pm\sqrt{25 - x^2}$: two functions
$x = y^2 - 4y$	already one function, of y

Reading how the variables change

- If a chord between nearby points has a positive slope (like +1 above), x and y increase together; if negative, one increases as the other decreases.
- The curve is horizontal where y stops changing and vertical where x stops changing.

Worked example

A boat is 500 m from a beacon at the origin ($x^2 + y^2 = 25$, in units of 100 m) and on the line $x = 4$. Where can it be?

- 1 Put $x = 4$ into the circle: $16 + y^2 = 25$, so $y^2 = 9$ and $y = 3$ or -3 .

(4, 3) or (4, -3), so give both

Test trap

- **Dropping the $\pm$.** $y^2 = 9$ gives $y = 3$ or -3 . Using only the top half of the circle loses (4, -3).
- **Writing down a point that doesn't exist.** On $x = 6$, $y^2 = -11$, and no real number squared is negative, so the line misses the circle: there is no point.

Quick check

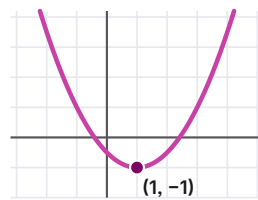
On the top half of $\frac{x^2}{4} + y^2 = 1$, does y rise or fall as x goes from 0 to 2? Where is the curve vertical?

Answer: It falls, from (0, 1) to (2, 0). It is vertical at $(\pm 2, 0)$.

4.6 Conic sections

In this course their axes of symmetry are horizontal or vertical

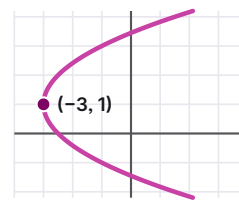
Parabola, opens up or down



$$y - k = a(x - h)^2$$

$$y + 1 = \frac{1}{2}(x - 1)^2: \text{vertex } (1, -1), \text{ opens up because } a > 0$$

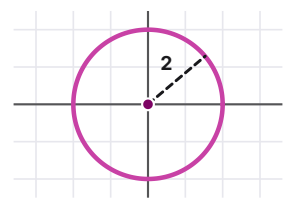
Parabola, opens left or right



$$x - h = a(y - k)^2$$

$$x + 3 = \frac{1}{2}(y - 1)^2: \text{vertex } (-3, 1), \text{ opens right because } a > 0$$

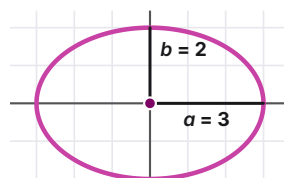
Circle



$$(x - h)^2 + (y - k)^2 = r^2$$

$$x^2 + y^2 = 4: \text{center } (0, 0), \text{ radius } 2$$

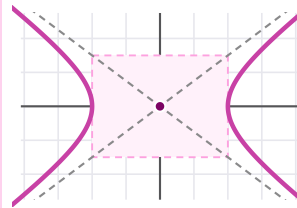
Ellipse



$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1: \text{it reaches 3 to each side and 2 up and down. It is a circle when } a = b.$$

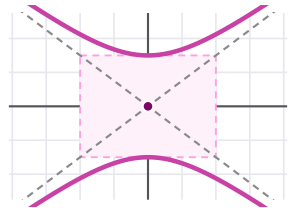
Hyperbola, opens left and right



$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

$$\frac{x^2}{4} - \frac{y^2}{2.25} = 1: a = 2, b = 1.5$$

Hyperbola, opens up and down



$$\frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1$$

$$\frac{y^2}{2.25} - \frac{x^2}{4} = 1: a = 2, b = 1.5, \text{ opening up and down because the } y^2 \text{ term is positive}$$

— = a hyperbola's asymptotes

□ the box, $2a$ wide and $2b$ tall, whose diagonals are the asymptotes

Squared terms in the equation

Squared terms in the equation	Conic
only one of x^2 and y^2	parabola
both, with equal coefficients	circle
both, same sign, different coefficients	ellipse
both, opposite signs	hyperbola

Some equations of this form graph as a single point, two crossing lines or nothing: $x^2 + y^2 + 1 = 0$ has no points at all.

Worked example

Put $9x^2 + 4y^2 - 18x + 16y - 11 = 0$ in standard form and read it.

- 1 Group the x terms and the y terms:
 $9(x^2 - 2x) + 4(y^2 + 4y) = 11$.
- 2 Complete each square. Adding 1 inside the x bracket adds $9 \cdot 1 = 9$, and adding 4 inside the y bracket adds $4 \cdot 4 = 16$, so add both on the right too:
 $9(x - 1)^2 + 4(y + 2)^2 = 11 + 9 + 16 = 36$.
- 3 Divide by 36: $\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{9} = 1$.

Ellipse, center (1, -2), reaching 2 to each side and 3 up and down

Test trap

- **Reading a as the denominator.** The denominator is a^2 , so $\frac{(x-1)^2}{4}$ means $a = 2$, and the curve reaches 2 units to each side of the center. Taking $a = 4$ draws it twice as wide.
- **Getting the center's signs backward.** $(x + 3)^2$ is $(x - (-3))^2$, so the center of $(x + 3)^2 + (y - 1)^2 = 4$ is $(-3, 1)$.
- **Using the bigger number to decide which way a hyperbola opens.** It opens along its positive squared term, so $\frac{y^2}{4} - \frac{x^2}{16} = 1$ opens up and down.
- **Flipping the asymptote slope.** The slopes are $\pm$ (square root of the number under y^2) $\div$ (square root of the number under x^2). In $\frac{y^2}{4} - \frac{x^2}{16} = 1$ that is $\pm \frac{2}{4} = \pm \frac{1}{2}$, not ± 2 .

Quick check

Name the conic, its center and its asymptotes:

$$4x^2 - 9y^2 + 8x + 36y - 68 = 0.$$

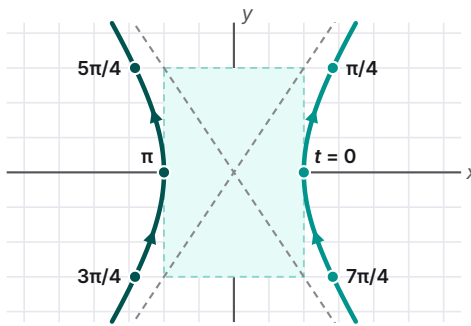
Answer: $4(x^2 + 2x) - 9(y^2 - 4y) = 68$, so $4(x + 1)^2 - 9(y - 2)^2 = 68 + 4 - 36 = 36$ and $\frac{(x+1)^2}{9} - \frac{(y-2)^2}{4} = 1$. It is a hyperbola opening left and right, with center $(-1, 2)$ and asymptotes $y - 2 = \pm \frac{2}{3}(x + 1)$.

4.7 Parametrization of implicitly defined functions

A parametrization is right if putting $x(t)$ and $y(t)$ into the equation makes it true for **every** t , and its points cover the whole curve.

Curve	$(x(t), y(t))$
$y = f(x)$	$(t, f(t))$
$x = g(y)$	$(g(t), t)$
inverse of f	$(f(t), t)$, if f is one-to-one
ellipse	$(h + a \cos t, k + b \sin t)$
hyperbola, opens left and right	$(h + a \sec t, k + b \tan t)$
hyperbola, opens up and down	$(h + a \tan t, k + b \sec t)$

Use $0 \leq t \leq 2\pi$. The hyperbolas skip $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, where $\sec t$ and $\tan t$ are undefined.



$(3 \sec t, 3 \tan t)$ draws $\frac{x^2}{9} - \frac{y^2}{9} = 1$, shown with its box and dashed asymptotes. The right branch comes from $\cos t > 0$, the left from $\cos t < 0$.

Worked example

Show that $(1 + 3 \cos t, -2 + 2 \sin t)$ traces $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$.

- 1 Substitute:

$$\frac{9 \cos^2 t}{9} + \frac{4 \sin^2 t}{4} = \cos^2 t + \sin^2 t = 1.$$
- 2 As t runs from 0 to 2π , $(\cos t, \sin t)$ goes all the way around, so the whole ellipse is traced.

It fits for every t and covers the whole curve ✓

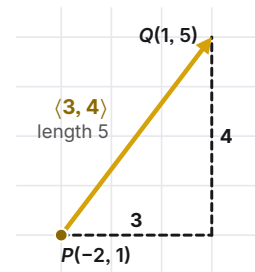
Test trap

- Checking one point instead of every t . $(3 \cos t, 2 \sin 2t)$ fits $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at $t = 0$, but at $t = \frac{\pi}{4}$ the left side is 1.5. Substitute a general t .
- Missing part of the curve. (t^2, t^4) fits $y = x^2$, but $x = t^2$ is never negative, so it draws only the right half; (t, t^2) draws it all.

4.8 Vectors

A vector has a length and a direction, and it can start at any point

From P to Q



P to Q $\langle x_2 - x_1, y_2 - y_1 \rangle$, end minus start

Length $\| \langle a, b \rangle \| = \sqrt{a^2 + b^2}$

Multiply by k $k \langle a, b \rangle = \langle ka, kb \rangle$: parallel, and reversed if $k < 0$

Sum add components, or start the second arrow at the tip of the first ("head to tail"), in either order (grey dashed)

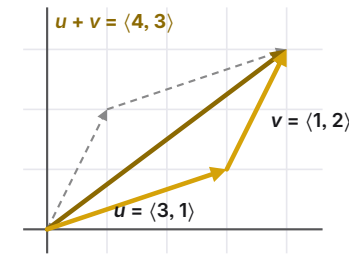
Unit vector $\vec{v}/\|\vec{v}\|$ has length 1 and points the same way as $\vec{v}$ (for $\vec{v} \neq \vec{0}$)

$\vec{i}$ and $\vec{j}$ $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$, so $\langle a, b \rangle = a\vec{i} + b\vec{j}$

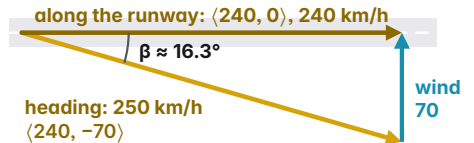
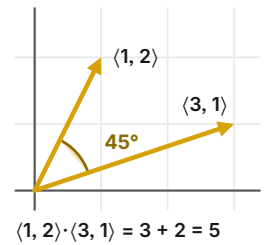
Length r , angle θ $\langle r \cos \theta, r \sin \theta \rangle$

Dot product $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 = \|\vec{u}\| \|\vec{v}\| \cos \theta$

Head to tail



Angle between two vectors



Worked example

A plane flies at 250 km/h through the air in a 70 km/h wind blowing straight across the runway. At what angle β to the runway must it point, and how fast does it go along it?

- 1 The sideways part of the plane's velocity must cancel the wind:
 $250 \sin \beta = 70$, so $\beta \approx 16.3^\circ$.
- 2 The part along the runway is
 $\sqrt{250^2 - 70^2} = \sqrt{57600} = 240$.

$\beta \approx 16.3^\circ$, at 240 km/h

Test trap

- Adding lengths. Lengths add only when vectors point the same way, so add components first:
 $\langle 250, 0 \rangle + \langle 0, 70 \rangle = \langle 250, 70 \rangle$, of length about 259.6, not 320.
- Subtracting in the wrong order. From $P(-2, 1)$ to $Q(1, 5)$ the vector is end minus start, $(1 - (-2), 5 - 1) = \langle 3, 4 \rangle$. Start minus end gives $\langle -3, -4 \rangle$, the opposite direction.

Testing for a right angle

Two nonzero vectors are perpendicular exactly when their dot product is 0, because $\cos 90^\circ = 0$. A positive dot product means an angle under 90° ; a negative one, over 90° .

A video-game guard's field of view

A video-game guard at $(2, 1)$ faces along $\langle 1, 0 \rangle$ and sees 30° to either side. The direction from the guard to you at $(5, 5)$ is $\langle 3, 4 \rangle$, unit vector $\langle 0.6, 0.8 \rangle$. The angle θ between the two directions has $\cos \theta = \langle 1, 0 \rangle \cdot \langle 0.6, 0.8 \rangle = 0.6$, so $\theta \approx 53^\circ$ and the guard can't see you.

Worked example

Find the angle between $\vec{u} = \langle 1, 2 \rangle$ and $\vec{v} = \langle 3, 1 \rangle$.

- 1 $\vec{u} \cdot \vec{v} = 1(3) + 2(1) = 5$, and the lengths are $\sqrt{5}$ and $\sqrt{10}$.
- 2 $\cos \theta = \frac{5}{\sqrt{5} \sqrt{10}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$

$\theta = 45^\circ$

Worked example

Find the components of a vector of length 10 at 150° (counterclockwise from the positive x -axis), and the unit vector in the direction of $\langle 6, -8 \rangle$.

- 1 $\langle 10 \cos 150^\circ, 10 \sin 150^\circ \rangle = \langle -5\sqrt{3}, 5 \rangle \approx \langle -8.66, 5 \rangle$
- 2 $\|\langle 6, -8 \rangle\| = 10$, so the unit vector is $\langle 0.6, -0.8 \rangle$.

$\langle -8.66, 5 \rangle$ and $\langle 0.6, -0.8 \rangle$

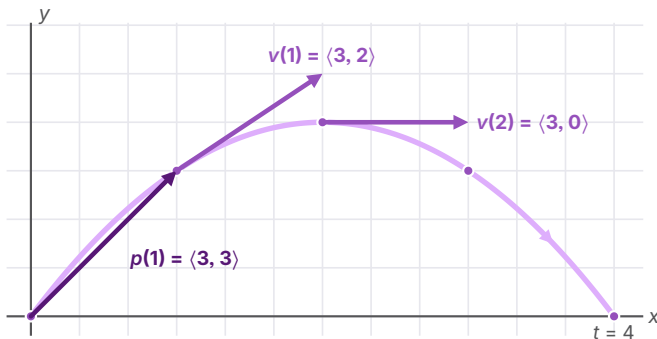
Quick check

Find k so that $\langle k, 3 \rangle$ is perpendicular to $\langle 6, -4 \rangle$.

Answer: $6k - 12 = 0$, so $k = 2$.

4.9 Vector-valued functions

The point $(x(t), y(t))$ can also be written as a position vector, $p(t) = \langle x(t), y(t) \rangle = x(t)\vec{i} + y(t)\vec{j}$, pointing from the origin to the particle.



A hopping probe at $p(t) = \langle 3t, 4t - t^2 \rangle$ meters. You are told its velocity: $v(t) = \langle 3, 4 - 2t \rangle$ m/s. The darkest arrow is the position vector $p(1)$, drawn from the origin; the other two arrows are velocities, which start at the probe and point along its path.

What you work out	What it tells you
$\ p(t)\ $	the distance from the origin
the sign of v 's first component	moving right (+) or left (-)
the sign of v 's second component	moving up (+) or down (-)
$\ v(t)\ $	the speed

Average velocity

$$\frac{p(t_2) - p(t_1)}{t_2 - t_1}$$

For the probe from $t = 0$ to 2: $\frac{(6, 4) - (0, 0)}{2} = \langle 3, 2 \rangle$ m/s, of length about 3.61 m/s. From 0 to 4 it is $\langle 3, 0 \rangle$, only 3 m/s: it measures the change in position, not the curved path flown.

Worked example

Where is the probe at $t = 1$, how far is it from the origin, which way is it going, and how fast?

- $p(1) = \langle 3, 3 \rangle$, which is $\sqrt{18} \approx 4.24$ m from the origin.
- $v(1) = \langle 3, 2 \rangle$: both components are positive, so it moves right and up, at $\sqrt{13} \approx 3.61$ m/s.

At $(3, 3)$, about 4.24 m from the origin, moving right and up at about 3.61 m/s

Test trap

- Treating $\|p\|$ as the distance traveled.** $\|p(t)\|$ is how far the particle is from the origin right now. A pod circling a space station on a 100 m ring travels about 628 m per lap, yet $\|p\|$ stays 100.
- Thinking the particle stops at the top.** At the top of the hop ($t = 2$), $v = \langle 3, 0 \rangle$. Only the up-and-down part is 0; the probe still moves right at 3 m/s.
- Reading direction from the position.** The signs of p say where the particle is; the signs of v say which way it goes. At $t = 3$ the probe is at $(9, 3)$, above the ground, but $v(3) = \langle 3, -2 \rangle$, so it is coming down.

4.10 Matrices

Each entry of a product is a row times a column

	energy	vocals	acoustic		Ava	Ben		Ava	Ben
Ember	3	1	0	$\times$	2	0	$=$	7	1
Canopy	0	2	3		1	1		2	8
Halo	1	3	1		0	2		5	5
	songs: 3×3				tastes: 3×2			scores: 3×2	

Matching songs to listeners at a silent disco. Each song (a row) is scored for energy, vocals and acoustic sound, and each listener's tastes (a column) weight those three. The shaded row (Canopy) times the shaded column (Ben) gives the shaded entry, $0 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 = 8$, Ben's best match. Ava's best is Ember, at 7.

Matrix sizes and products

An $m \times n$ matrix has m rows and n columns. A product AB exists only when A has as many columns as B has rows (the underlined inside sizes match). The answer has A 's number of rows and B 's number of columns:

$$(m \times \underline{n})(\underline{n} \times p) = m \times p$$

The entry in row i , column j of AB is row i of A times column j of B : multiply matching entries and add.

Worked example

Multiply both ways: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

$$\begin{aligned} 1 \quad AB &= \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \\ 2 \quad BA &= \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix} \end{aligned}$$

$AB \neq BA$, so the order matters

Which products exist

Sizes multiplied	Size of the answer
$(2 \times 3)(3 \times 4)$	2×4
$(3 \times 4)(2 \times 3)$	none, because $4 \neq 2$
$(2 \times 2)(2 \times 1)$	2×1 : a matrix times a vector
$(1 \times 3)(3 \times 1)$	1×1 : a dot product

Test trap

- Multiplying entry by entry.** Each entry of a product is a row times a column, so multiplying the matching entries gives the wrong answer:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \text{ not } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Assuming BA exists because AB does.** A 2×3 times a 3×1 works and gives 2×1 , but a 3×1 times a 2×3 doesn't, because $1 \neq 2$.

Quick check

Compute $\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. What size is a $(3 \times 2)(2 \times 5)$ product?

Answer: $\begin{bmatrix} 8 - 5 \\ 0 + 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$; 3×5 .

4.11 The inverse and determinant of a matrix

One number, the determinant $ad - bc$, tells you whether a 2×2 matrix can be undone and how much it scales area.

The determinant and the inverse

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

Determinant $\det A = ad - bc$

Inverse $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, only if $\det A$ is not 0

Identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $AA^{-1} = A^{-1}A = I$

To invert, swap a and d , change the signs of b and c , and divide every entry by the determinant.

Area from the determinant

A matrix with columns $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ turns the unit square into this parallelogram. The 4×3 box has area 12, and the grey pieces, marked with their areas, add up to 7, so the area is $12 - 7 = 5 = 3 \cdot 2 - 1 \cdot 1$.

The area is $|\det A|$; a negative sign means the picture is also flipped. A determinant of 0 flattens the square onto a line.

Worked example

A recording mixes two tracks: left = $2 \times$ drums + bass and right = drums + bass. If the left speaker plays at level 5 and the right at 3, how loud is each track?

$$1 \quad A \begin{bmatrix} \text{drums} \\ \text{bass} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \text{ with } A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and } \det A = 1, \text{ so}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

$$2 \quad \text{Multiply by } A^{-1}: \begin{bmatrix} \text{drums} \\ \text{bass} \end{bmatrix} = A^{-1} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 - 3 \\ -5 + 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Drums 2, bass 1. Check: left = $2(2) + 1 = 5$ ✓

Test trap

– **Forgetting to divide by the determinant.** Swapping and changing signs in N (below) without dividing gives 10 times the inverse. Divide every entry by $\det N = 12 - 2 = 10$:

$$N = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}, N^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.3 & -0.1 \\ -0.2 & 0.4 \end{bmatrix}$$

– **Hunting for an inverse when the determinant is 0.** $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ sends both $(2, 1)$ and $(1, 2)$ to $(3, 6)$, so nothing can undo it. Check the determinant first; if it is 0, write "no inverse".

Quick check

$$\text{Invert } \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Answer: the determinant is $6 - 5 = 1$, so the inverse is $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$.

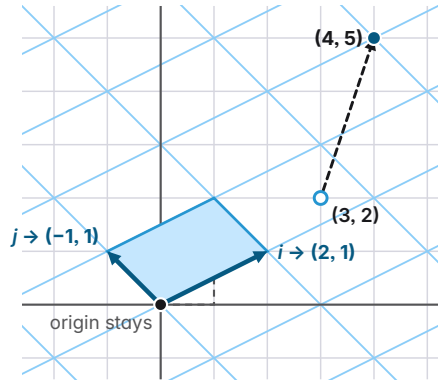
4.12 Linear transformations and matrices

A linear transformation moves every point of the plane by one rule, $L(\vec{v}) = A\vec{v}$, where each new coordinate has the form $ax + by$.

A rule and its matrix

$L(x, y) = (ax + by, cx + dy)$ has the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Row 1 holds the numbers for the new x , and row 2 the numbers for the new y . The origin never moves, because $A\vec{0} = \vec{0}$.



A city map after the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ acts on it.

The grey square grid is the map before and the blue slanted grid is after: the streets stay straight, parallel and evenly spaced. The square block at the origin becomes a parallelogram of area 3, the determinant, and the café at $(3, 2)$ (hollow dot) moves to $(4, 5)$.

Worked example

Move all four corners of the unit square with $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ at once, by writing the corners as the columns of one 2×4 matrix.

$$A \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

New corners $(0, 0)$, $(2, 1)$, $(1, 2)$, $(-1, 1)$

Rule	Linear?	Why
$(2x - y, x + y)$	yes	both parts have the form $ax + by$
$(3x, 0)$	yes	it flattens the plane onto the x -axis, but still has that form
$(x + 1, y)$	no	it slides the origin to $(1, 0)$
(x^2, y)	no	it keeps the origin, but x^2 isn't $ax + by$

Test trap

- **Writing the vector as a row.** The vector goes on the right as a column: $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$. Putting the row $[-1 \ 3]$ on the left gives $(1, 4)$, which is wrong.
- **Calling a rule linear because it keeps the origin.** (x^2, y) keeps the origin and isn't linear. Moving the origin proves a rule is not linear; to prove it is, write it as $(ax + by, cx + dy)$.

Quick check

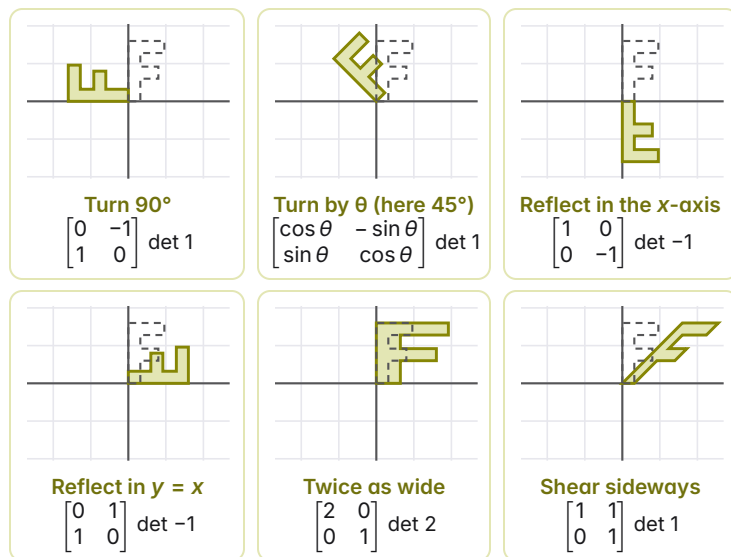
Write the matrix of $L(x, y) = (x - 4y, 3x)$. Separately, find

$$\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Answer: $\begin{bmatrix} 1 & -4 \\ 3 & 0 \end{bmatrix}$; $\begin{bmatrix} 6 - 5 \\ 4 + 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 24 \end{bmatrix}$.

4.13 Matrices as functions

The columns of a matrix are where $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ land. Every turn here is counterclockwise.



-- the letter F before the F after the matrix acts

Combining and undoing transformations

- To do B first and then A , multiply by AB : the matrix next to the vector acts first, as in $A(B\vec{v})$.
- To undo both, undo the last one first: $(AB)^{-1} = B^{-1}A^{-1}$ (shoes come off before socks).
- Every area is multiplied by $|\det A|$, so making a shape 3 times wider and 3 times taller multiplies its area by 9, not 3. A negative determinant also mirrors the picture.

Worked example

A transformation sends $\vec{i}$ to $(2, 1)$ and $\vec{j}$ to $(1, 2)$. Find its matrix, where it sends $(2, 1)$, and how it changes area.

- 1 The images are the columns: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and $A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

$\det A = 4 - 1 = 3$, so every area triples

Test trap

- Multiplying in the order you read.** "Do B , then A " is AB , not BA . With R the 90° turn and D the doubling of width, DR turns first and RD stretches first:

$$DR = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, RD = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

- Putting $-\sin \theta$ in the wrong corner.** For a counterclockwise turn it sits top right. Test with $\theta = 90^\circ$: the matrix must send $\langle 1, 0 \rangle$ to $\langle 0, 1 \rangle$.

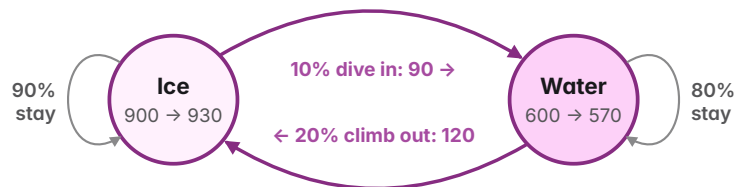
Quick check

Find the matrix that reflects in the y -axis and then turns 90° counterclockwise. Which single transformation is it?

Answer: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, which sends (x, y) to $(-y, -x)$: a reflection in the line $y = -x$.

4.14 Matrices modeling contexts

In a transition model, the same share of each group, or state, moves at every step, as in this made-up penguin colony.



Setting up and running the model

Write the state as a column with the ice count on top. Each column of M says where one group goes, so it adds up to 1.

$$M = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

One step later	$\vec{s}_1 = M\vec{s}_0$
n steps later	$\vec{s}_n = M^n\vec{s}_0$
One step earlier	$\vec{s}_0 = M^{-1}\vec{s}_1$
Steady state	$M\vec{s} = \vec{s}$

Worked example

There are 900 penguins on the ice and 600 in the water. Find the numbers for the next two hours.

- 1 $M \begin{bmatrix} 900 \\ 600 \end{bmatrix} = \begin{bmatrix} 0.9(900) + 0.2(600) \\ 0.1(900) + 0.8(600) \end{bmatrix} = \begin{bmatrix} 930 \\ 570 \end{bmatrix}$
- 2 Multiplying $(930, 570)$ by M the same way gives $(951, 549)$.

The total stays 1500 every hour

Worked example

Find the steady state, and the numbers one hour before $(930, 570)$.

- 1 With x on the ice and y in the water, $0.9x + 0.2y = x$ gives $x = 2y$. With $x + y = 1500$: $y = 500$, $x = 1000$.
- 2 $\det M = 0.72 - 0.02 = 0.7$, so $M^{-1} = \frac{1}{0.7} \begin{bmatrix} 0.8 & -0.2 \\ -0.1 & 0.9 \end{bmatrix}$, which sends $(930, 570)$ back to $(900, 600)$.

Steady state: 1000 on the ice and 500 in the water

Test trap

- Thinking the steady state means nobody moves.** At $(1000, 500)$, 100 penguins move each way every hour; only the counts stay put.
- Assuming every model settles down.** If everyone switches sides at every step, $(600, 400)$ and $(400, 600)$ take turns forever. Run a few steps to check.
- Running the model too far back.** M^{-1} undoes one hour, but seven hours before $(900, 600)$, M^{-7} predicts about -214 penguins on the ice, which is impossible. Trust it only near the times it was built from.

Quick check

Each day 30% of station A's bikes move to B and 10% of B's move to A. With 400 bikes in all, find the steady state.

Answer: with a bikes at A and b at B, $0.3a = 0.1b$ and $a + b = 400$: $a = 100$, $b = 300$.

What to write for full credit on a Unit 4 test

When the question says	What to write
4.1 "Sketch the path"	Make a table of t , x and y , join the points in order of increasing t , add arrows for the direction, and label a dot at each end.
4.2 "Where does it cross the y -axis?"	Solve $x(t) = 0$, then give the point $(0, y(t))$ for each solution, not just the value of t .
4.3 "Which way is it moving at $t = 2$?"	Say whether $x(t)$ and $y(t)$ are increasing or decreasing at $t = 2$ (compare $t = 1.9$ and 2.1), then give the direction in words.
4.3 "Find the average rates of change"	Work out $\Delta x/\Delta t$ and $\Delta y/\Delta t$ with units; divide them for the chord's slope, $\Delta y/\Delta x$.
4.4 "Parametrize the circle" or "the segment"	Write $x(t)$, $y(t)$ and the interval for t , then check that $t = 0$ gives the start and the direction is correct.
4.5 "Write y as a function of x "	Give every piece of the curve (keep the $\pm$ when it appears), each with its domain.
4.6 "Identify and sketch the conic"	Show the completed squares, the center, a and b (the square roots of the denominators), which way it opens, and any asymptotes.
4.7 "Show the parametrization works"	Substitute and simplify to 1 with $\cos^2 t + \sin^2 t = 1$ (or $\sec^2 t - \tan^2 t = 1$), then say why it covers the whole curve.
4.8 "Find the angle between the vectors"	Show the dot product and both lengths, then $\cos \theta$, then θ in the units the question uses.
4.9 "How fast?" or "How far from the origin?"	Use $\ v(t)\ $ for the speed and $\ p(t)\ $ for the distance from the origin, with units.
4.10 4.11 "Find AB " or "Find A^{-1} "	Check the sizes first. For an inverse, show the determinant and say it isn't 0, then check that $AA^{-1} = I$.
4.12 4.13 "Find the matrix of the transformation"	Find where $\vec{i}$ and $\vec{j}$ land and write those as the columns. For two transformations, put the first one on the right.
4.14 "Predict the next or an earlier state"	Label the states, check that each column adds up to 1, then multiply by M or M^{-1} . The total must not change, and no count can be negative.

Graph a parametric path

TI-84 Plus CE. Press **mode** and highlight PARAMETRIC on the row FUNCTION. Press **enter** on each. In **Y=**, type $x(t)$ after $X1T=$ and $y(t)$ after $Y1T=$ (the **x,t,θ,n** key types T). In **window**, set Tmin and Tmax to your interval and Tstep to about 0.05, then press **graph**.

TI-Nspire CX. On a Graphs page, press **menu** → Graph Entry/Edit → Parametric. Type $x_1(t)$ and $y_1(t)$, set the interval for t on the same line, and press **enter**.

Desmos. Type the point with t in it, such as $(\cos(t), \sin(t))$. Desmos first draws it for $0 \leq t \leq 1$; change the two numbers under the expression to your interval, such as 0 and 2π .

Work with matrices

TI-84 Plus CE. Press **2nd** **x⁻¹** (MATRIX), go to EDIT and choose [A]. Type the size, then the entries. On the home screen, insert [A] from MATRIX → NAMES. [A]×[B] multiplies, [A]**x⁻¹** inverts and MATRIX → MATH → det(gives the determinant. ERR:DIM MISMATCH means the sizes don't allow the product.

TI-Nspire CX. On a Calculator page, choose **menu** → Matrix & Vector → Create → Matrix, set the size, and type the entries. Store it with **ctrl** **var** (sto→) as a , and make b the same way. Then type $a*b$ to multiply, a^{-1} to invert and $\det(a)$ for the determinant.

Desmos. The graphing calculator has no matrices, so use the separate Desmos Matrix Calculator: make each matrix with New Matrix, multiply with $A*B$, and find inverses and determinants on its keypad.

Solve $x(t) = 0$ to find a crossing

TI-84 Plus CE. In function mode (FUNC on the **mode** screen), type $x(t)$ in Y1 with X in place of t , then **graph**. Press **2nd** **trace** (CALC) and choose 2: zero. Move the cursor left of the crossing and press **enter**, move it right of the crossing and press **enter**, then press **enter** once more. The X shown is the time; put it into $y(t)$ to get the point.

TI-Nspire CX. On a Graphs page, type $x(t)$ as $f1(x)$ with x in place of t . **menu** → Analyze Graph → Zero, then click left of the crossing and right of it.

Desmos. Type $x(t)$ with x in place of t , such as $y = x^2 - 4x$, and tap the grey dots where the graph meets the axis.

Checks before you hand in a test

- Does every parametrization have its interval?
- Did I keep every $\pm$?
- Is the first transformation on the right?
- Do lengths, speeds and areas have units?

The notation

$(x(t), y(t))$	where the particle is at time t
$\langle a, b \rangle$	a vector: a across and b up, from any starting point
$\ v\ $	its length; the speed if v is a velocity
$\vec{u} \cdot \vec{v}$	the dot product, one number
$\det A$	a single number; $ \det A $ is how much A scales area
A^{-1}	the matrix that undoes A
$M^n s_0$	the state n steps after s_0

Where Unit 4 comes back

- AP Calculus BC** has a unit on parametric equations, polar coordinates and vector-valued functions. There you find the velocity yourself, by differentiating $x(t)$ and $y(t)$.
- Physics** adds forces and velocities head to tail and splits them into parts with $r \cos \theta$ and $r \sin \theta$.
- Computer graphics** uses matrices to move every point of a picture at once, like the letter F on page 8.
- Transition models** are a first look at what college courses call Markov chains.